# A Theoretical Random-Error Propagation Law for Anblock-Networks With Constrained Boundary

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# A Theoretical Random-Error Propagation Law for Anblock-Networks With Constrained Boundary

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*Zusammenfassung*

Anblocknetze setzen sich aus einer großen Anzahl aneinanderliegender elementarer Figuren zusammen. Für jede Figur liegen separate photogrammetrische Messungen vor, die es gestatten, die Gestalt der Figur in einem unbekannten Maßstab zu rekonstruieren. Ebner hat für große idealisierte Anblocknetze mit festgehaltenen Randpunkten mittels Computer numerische Fehlerstudien durchgeführt. An Hand seiner Resultate vermutete er, daß der durchschnittliche Koordinatenfehler nach strengem Ausgleich einem ähnlichen logarithmischen Gesetz folgend mit der Anzahl der Netzpunkte anwächst, wie dies vom Autor im Falle großer Nivellementnetze nachgewiesen wurde. Ebners Vermutung wird hier auf analytischem Wege bewiesen. Die genaue Form des asymptotischen Fehlergesetzes wird gefunden.

*Summary*

Anblock-networks are composed of a large number of adjacent elementary figures. For each figure separate photogrammetric measurements are taken, allowing to reconstruct the shape of the figure in an unknown scale. Ebner has performed computer simulation studies for large idealized

Anblock-networks with fixed boundary points. From his numeiical results he conjectured that the average r. m. s. coordinate error after rigorous adjustment increases with the number of network points according to a logarithmic law, similar to one discovered by the author in the case of leveling networks. Ebners conjecture is pioved correct by an analytical argument. The precise form of the asymptotic error propagation law is exhibited.

## 1. *Introduction*

In Meissl (1971) a theoretical error propagation model for photogrammetric block-networks with fixed boundary points was presented. This model reflects in a satisfactory way the random error propagation of the height errors. In a quadratic block of $n^2$ models a suitable defined average height error grows like a constant times $n$. As far as horizontal errors are concerned, the above mentioned error propagation model is not satisfactory.

F. Ackermann and H. Ebner recently suggested to theoretically investigate so-called Anblock-networks in the plane. Computer simulation studies by Ebner (1970) for up to $n = 100$ indicated a very slow growth of the point errors in such networks. Ebners conjecture was that a similar error-propagation law might hold, as was found in Meissl (1970) for idealized leveling networks. In the latter study an asymptotic growth of a certain average error like a constant times $\sqrt{\log_e (n^2)}$ was proved. In this study we shall show that Ebners conjecture is correct. The precise form of the error propagation law will be exhibited. Proofs of mathematical details will not be given, because they are essentially the same as in Meissl (1970).

## 2. *A regular Anblock-network*

Consider the following idealized photogrammetric triangulation problem. The preliminary position of $(m + 1)(n + 1)$ points in the plane forms a grid as shown in the following figure[1]). The elementary rectangles of the grid have side lenghts 2a, 2b. For each of the elementary rectangles we have a model showing the position of the four corner-points in an unknown scale. This is equivalent to a complete angular information about the configuration formed by the quadruple of points belonging to a model. The unknown scale of the models shall be approximately the same for all rectangles, and shall be approximately 1:1 to reality. The latter assumption does not restrict generality.

In each of the models the model-coordinates of the four points are measured. The model coordinate system has its origin in the center of each model. Since the approximate scale of the models is 1:1, the measured model coordinates have values approximately equal to $\pm a$, $\pm b$. The r. m. s. error of a model coordinate measurement in $x$ or $y$ shall be 1. If this error is different of 1, but still the same for all measurements, then the final formulas have to be augmented by a simple factor. Zero correlation between all measurements is assumed.

All boundary points of the grid are assumed fixed. The position of the interior points of the grid shall be determined from the model measurements. Adjustment shall be rigorous.

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[1]) The figure has been set, by technical causes on page 73.

### 3. *Forming the normal equations*

We briefly review the formation of the normal equations which have been used in Ebners study. Adjustment according to variation of parameters is used. The unknown parameters are

(1) the coordinates $x_{ij}$, $y_{ij}$, $i = 1, \ldots, m-1, j = 1, \ldots, n-1$, of the inner networks points,

(2) two translation-, one rotation-, and one scale parameter for each model.

As translation parameters for the k-th model we take the unknown position $\xi_k$, $\eta_k$ of the model-center in the network coordinate system. The rotation parameter $\varphi_k$ is the angle by which the model coordinate system is rotated with respect to the network system. The scale parameter $\mu_k$ refers to the unknown model-scale.

Approximate values for the coordinates are in an obvious way obtained from the figure. For $\xi_k$, $\eta_k$ the centers of the rectangles are taken, while $\varphi_k = 0$, $\mu_k = 1$ holds approximately for all models.

The linearized observation equation for the k-th model are

$$\delta x_k^{(r)} + v_k^{(r)} = \Delta x_k^{(r)} - \Delta \xi_k + (y_k^{(r)} - \eta_k)\,\Delta \varphi_k - (x_k^{(r)} - \xi_k)\,\Delta \mu_k$$
$$\delta y_k^{(r)} + w_k^{(r)} = \Delta y_k^{(r)} - \Delta \eta_k - (x_k^{(r)} - \xi_k)\,\Delta \varphi_k - (y_k^{(r)} - \eta_k)\,\Delta \mu_k \qquad r = 1, 2, 3, 4$$

Explanation of symbols: $\delta x_k^{(r)}$, $\delta y_k^{(r)}$: deviation of measured model coordinates of model point No. $r$ from approximate values $\pm a$, $\pm b$. $v_k^{(r)}$, $w_k^{(r)}$: corrections to model coordinate measurements. $\Delta x_k^{(r)}$, $\Delta y_k^{(r)}$: variation of the coordinates $x_k^{(r)}$ $y_k^{(r)}$ of the network point corresponding to model point No. $r$. $\Delta \xi_k$, $\Delta \eta_k$: variation of model center. $\Delta \varphi_k$, $\Delta \mu_k$: variation of model orientation and scale.

In the usual way one forms normal equations for each model. The weight matrix is the identity. The parameters $\Delta \xi_k$, $\Delta \eta_k$, $\Delta \varphi_k$, $\Delta \mu_k$ are eliminated by a standard procedure. The normal equations of all models are summed to form the final normal equations. It is seen that the problem completely decomposes into an $x$ and $y$ problem with identical normal equation matrix $N$. $N$ is implied by

$$-\alpha \Delta x_{i-1,j} - \beta \Delta x_{i,j-1} + 2 \Delta x_{i,j} - \beta \Delta x_{i,j+1} - \alpha \Delta x_{i+1,j} = \text{right hand side}$$

$$\Delta x_{i,o} = \Delta x_{i,n} = \Delta x_{o,j} = \Delta x_{m,j} = 0$$

$$i = 1, \ldots, m-1, \; j = 1, \ldots, n-1$$

Thereby we have denoted:

$$\alpha = \frac{1 + \bar{a}^2 - \bar{b}^2}{2}, \; \beta = \frac{1 - \bar{a}^2 + \bar{b}^2}{2},$$

with

$$\bar{a} = a/\sqrt{a^2 + b^2}, \; \bar{b} = b/\sqrt{a^2 + b^2}.$$

We have described the $x$-problem. The formally identical $y$-problem is obtained replacing $\Delta x_{ij}$ by $\Delta y_{ij}$.

The normal equation matrix $N$ is inverted, yielding the covariance matrix $M = N^{-1}$ of the adjusted coordinates.

## 4. *Results*

An analytical formula for $M$ can be given. It is obtained by a slight modification of the equations in Varga (1962), section 6.1. The formula for the element in the row No. $i, j$, column No. $p, q$ of $M$ is given by

$$m_{i,j;p,q} = \frac{4}{mn} \sum_{r=1}^{m-1} \sum_{s=1}^{n-1} \frac{1}{\lambda_{r,s}} \sin\frac{i r \pi}{m} \sin\frac{j s \pi}{n} \sin\frac{p r \pi}{m} \sin\frac{q s \pi}{n},$$

$$\lambda_{r,s} = 4\,\alpha \sin^2\left(\frac{r\pi}{2m}\right) + 4\,\beta \sin^2\left(\frac{s\pi}{2n}\right).$$

Form the trace of $M$, i. e.

$$tr(M) = \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} m_{i,j;i,j}.$$

Define an average error measure $\tau$ by

$$\tau = \sqrt{\frac{tr(M)}{(m-1)(n-1)}}.$$

The asymptotic behavior of $\tau$ is given by:

$$\tau = \sqrt{\frac{1}{4\pi\sqrt{\alpha\beta}}}\sqrt{\log_e(mn)} + 0\left(\frac{1}{\sqrt{\log_e(mn)}}\right),$$

$$\text{for } m \to \infty,\ n \to \infty \text{ and } c_1 \leq \frac{m}{n} \leq c_2$$

with some positive constants $c_1$, $c_2$.

Of course, any other asymptotic error information can be obtained from the formula for $M$. The asymptotic behavior of the r. m. s. error in $x$ or $y$ of the midpoint of the network is for example also given by that of $\tau$.

## 5. *Conclusion*

The essential result appears to be that the error propagation in these Anblock-networks has a very slow growth rate. Similar slow growth rates have been observed for the horizontal errors in photogrammetric block triangulation networks with strong boundary control, Brown (1971). It is therefore tempting to assume that the above logarithmic error propagation law also holds for bundle block networks. This, however, should be proved rigorously.

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# Analytische Behandlung einiger Grundaufgaben der Zweimedien-Photogrammetrie

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## 1. *Einleitung*

Bei der herkömmlichen Definition der Photogrammetrie wird normalerweise vorausgesetzt, daß die Projektionsstrahlen bei der Aufnahme außerhalb des Kameraobjektives nur ein Medium durchlaufen (Einmedien(EM)-Photogrammetrie). Erweitert man diese Definition auf mehrere Medien, so erhält man den Begriff der Mehrmedien(MM)-Photogrammetrie. Ihr einfachster Spezialfall ist die Zweimedien (ZM)-Photogrammetrie. Sie unterscheidet sich von der EM-Photogrammetrie dadurch, daß das Meßbild nur den Strahlengang bis zur Grenzfläche der beiden Medien vermittelt. Über den Verlauf der gebrochenen Strahlen kann erst bei Kenntnis der Lage und Form dieser Trennfläche sowie der optischen Eigenschaften der beiden Medien eine Aussage gemacht werden.

Die einfachste Grenzfläche ist eine Ebene. Bei ihr treten zu den den Strahlenverlauf bestimmenden Größen gegenüber der EM-Photogrammetrie 4 weitere Parameter hinzu: 3 Parameter, die die Lage der Trennebene festlegen und ein Parameter, der die optischen Eigenschaften des zweiten Mediums relativ zum ersten angibt (Brechungsindex).

## 2. *Das Abbildungsgesetz*

Das Abbildungsgesetz stellt die mathematischen Beziehungen zwischen den Objektpunkten $P$ und den Bildpunkten $P'$ dar (Fig. 1).

Es kann in der Form der Funktion

$$\mathbf{p}_0' = f(\mathbf{x}) \qquad \dots (1)$$

ausgedrückt werden. Diese enthält neben den Variablen der Objektkoordinaten $\mathbf{x}$ die Konstanten der Abbildung. Es sind dies die 6 Daten der äußeren Orientierung,