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Anharmonic Analysis by Collocation Method

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Anharmonic Analysis by Collocation Method

Von *Josef Kabeláč,* Praha und *Karel Večeře,* Pardubice

1. Introduction

The paper presents first an amplified collocation method on several signals and deals also with a free collocation. Attention is given to the analytical determination of correlation funciton and of cross correlation functions. Further the collocation method with several signals is applied, as well as the analytical correlation function to anharmonic analysis.

The paper links up with [1] and [2] and is working in the field of the publications [3] and [4], but on the base of totally different approach. The studies preceding the presented paper are mentioned in [5] and [6].

2. Collocation with one signal, analytical investigation of correlation function and applications

The fundamental equation of the *fixed* collocation method with one signal is, see [1] and [2],

$$\mathbf{Ax} + \mathbf{US} = \mathbf{l} + \mathbf{v}, \quad \mathbf{Q}_l, \tag{1}$$

where **A** is Jacobi's matrix of known partial derivatives,
x is denoting the unknown trend's vector,
U is an auxiliary matrix compiled from unit and null submatrices,
S the unknown vector of the signals,
v the unknown vector of random corrections,
l the known vector of absolute terms,
$\mathbf{Q}_l$ is the known matrix of cofactors between the mediating quantities.

The expression

$$\mathbf{v}^T\mathbf{Q}^{-1}\mathbf{v} + \mathbf{S}^T\mathbf{Q}_{ss}^{-1}\mathbf{S} - 2\mathbf{K}^T(\mathbf{Ax} + \mathbf{US} - \mathbf{l} - \mathbf{v})$$

is minimum only then when it holds true

$$\begin{aligned}
\mathbf{x} &= [\mathbf{A}^T(\mathbf{Q}_l + \mathbf{Q}_{ss})^{-1}\mathbf{A}]^{-1}\,\mathbf{A}^T(\mathbf{Q}_l + \mathbf{Q}_{ss})^{-1}\mathbf{l}, \\
\mathbf{K} &= (\mathbf{Q}_l + \mathbf{Q}_{ss})^{-1}\,(\mathbf{l} + \mathbf{Ax}), \\
\mathbf{v} &= -\mathbf{Q}_l\,\mathbf{K}, \\
\mathbf{s} &= \mathbf{Q}_{ss}\,\mathbf{K}, \\
\mathbf{s}_p &= \mathbf{Q}_{sps}\,\mathbf{K},
\end{aligned} \tag{2}$$

where **K** is the vector of the correlates and $\mathbf{s}_p$ the vector of predicted signals. The matrix $\mathbf{Q}_{SS}$ is a total covariance matrix and $\mathbf{Q}_{ss}$, $\mathbf{Q}_{sps}$ are covariance matrices between the signals on supporting and predicted points. In case that the condition of minimum between the elements of the unknown vector **x** occur, there can be spoken about the *free* collocation. The expression

$$\mathbf{x}^T\mathbf{Q}_{xx}^{-1}\mathbf{x} + \mathbf{v}^T\mathbf{Q}^{-1}\mathbf{v} + \mathbf{S}^T\mathbf{Q}_{ss}\mathbf{S} - 2\mathbf{K}^T\,(\mathbf{Ax} + \mathbf{US} - \mathbf{l} - \mathbf{v})$$

is minimum only then, when it is valid

$$\mathbf{K} = (\mathbf{A}\mathbf{Q}_{xx}\mathbf{A}^T + \mathbf{Q}_l + \mathbf{Q}_{ss})^{-1}, \tag{3}$$
$$\mathbf{x} = \mathbf{Q}_{xx}\mathbf{A}^T\mathbf{K},$$

and $\mathbf{v}$, $\mathbf{s}$, $\mathbf{s}_p$ see the equation (2). $\mathbf{Q}_{xx}$ is the matrix of cofactors between the unknown quantities of the vector $\mathbf{x}$. The estimate of the mean errors is similar to the procedure in (1).

Let us now pay attention to the main problem of this paragraph, to the analytical determination of the correlation function. Let us have a certain known function f(x) that is supposed to be continuous in the interval (— ∞, + ∞). Let it be true for the signal s,

$$s = f(x), \tag{4}$$

which is a periodical function with a period $<0, x_\pi>$. Further we have the arguments x_i and $x_i + d_{ij}$. Covariance and variance are

$$\mathrm{cov}_{ij} = \frac{1}{n}\sum_{i=1}^{n} f(x_i)f(x_i + d_{ij}),$$

$$\mathrm{var}_{ij} = \frac{1}{n}\sum_{i=1}^{n} f(x_i)f(x_i),$$

where $n = x_\pi / dx$. Then

$$\mathrm{cov}_{ij} = \lim_{dx\to 0}\frac{dx}{x_\pi}\sum_{i=1}^{n} f(x_i)f(x_i + d_{ij}) = \frac{1}{x_\pi}\int_0^{x_\pi} f(x)f(x + d_{ij})\,dx,$$

$$\mathrm{var}_{ij} = \lim_{dx\to 0}\frac{dx}{x_\pi}\sum_{i=1}^{n} f(x_i)f(x_i) = \frac{1}{x_\pi}\int_0^{x_\pi} f(x)f(x)\,dx.$$

The elements of the correlation matrix K_{ss} are given by the correlation function $K_{ij} = \mathrm{cov}_{ij} / \mathrm{var}_{ij}$ and thus

$$K_{ij} = \int_0^{x_\pi} f(x)f(x + d_{ij})\,dx \Big/ \int_0^{x_\pi} f(x)f(x)\,dx. \tag{5}$$

The correlation matrix $\mathbf{K}_{ss}$ will be used for computations by the collocation method after a little arrangement of the equation (2). The expression $\mathbf{Q}_l + \mathbf{Q}_{ss}$ changes into the form

$$\mathbf{K}_n = \alpha\,\mathbf{Q}_l + \mathbf{K}_{ss}, \tag{6}$$

where α is a certain coefficient, the tasks of which are (see also the Table 2):

a) to influence the rate of smoothing,
b) to get the matrix $\mathbf{Q}_l$ and $\mathbf{K}_{ss}$ into a numerical coincidence and
c) to increase the value of the determinant of the matrix $\mathbf{K}_n$ at model applications and to enable the inversion of the matrix.

Similar tasks are valid for the free collocation, equation (3). In model applications, in connection with the equation (4), there was applied the periodical function

$$s = A \sin (Bx + C), \tag{7}$$

where A presents a random and sin (Bx + C) a deterministic quantity.

As there the periodical functions are concerned, let us do now a linear transformation of the independent variable x into the interval $<0,2\pi>$, in which the given problem

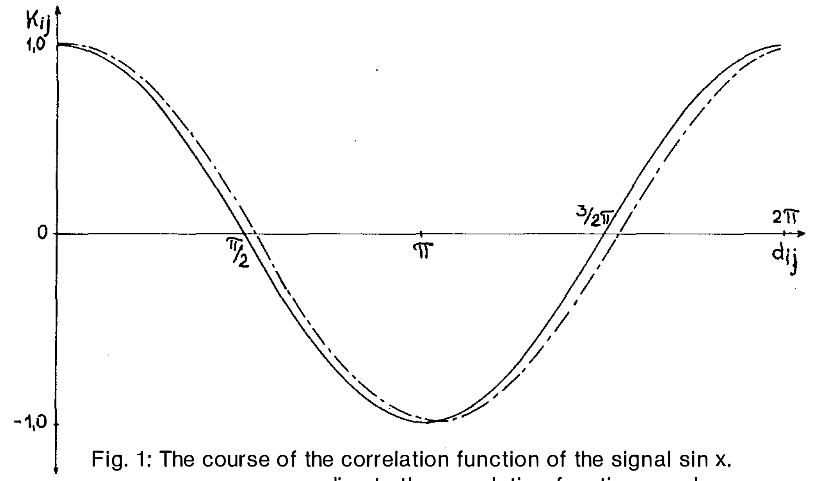

Fig. 1: The course of the correlation function of the signal sin x.
———— according to the correlation function cos d_{ij}
— — — — on base of numerical calculation

will be solved. After inserting the equation (7) into the equation (5), the latter one changes into a correlation function of the form

$$K_{ij} = \frac{A^2 \int_0^{2\pi} \sin(Bx + C) \sin[B(x + d_{ij}) + C]\, dx}{A^2 \int_0^{2\pi} \sin^2(Bx + C)\, dx} = \cos B d_{ij}, \tag{8}$$

where $d_{ij} = x_j - x_i$ and there the questionable random quantity A does not occur any more.

Using the relation (8) for different types of signals f(x) we get in the Table 1 correlation functions K_{ij} determined in analytical way.

The differences between analytical and numerical calculations are caused apparently on base of the fact that the number of random quantities used in the numerical calculation is limited.

Another numerical proving was realized in using the correlation function directly in the method of fixed collocation with one signal. A simplified model, see equation (2), has included: $\mathbf{A} = 0 \vee$ constant, $\mathbf{v} = \mathbf{0}$, $s = A \sin(Bx + C)$, $\mathbf{Q}_i = \mathbf{E}$. Further there were chosen on the interval $<0,2\pi>$ 18 supporting points ununiformly placed and 10 predicted points. To judge the true accuracy Δs the following relation was chosen:

$$\Delta s = \frac{1}{n} \sum_{i=1}^{n} | s_{exact\,i} - s_{colloc\,i} |, \qquad n = 18,$$

where $s_{exact\,i}$ is the accurate value and $s_{colloc\,i}$ is the value computed by the collocation method with the presented collocation function, see Table 2.

On base of the mentioned considerations and applications the following conclusions can be formulated.

1. The correlation function is a certain illustration of the signal function, where in the correlation function only the frequency of the signal function appears.
2. The correlation function is invariant towards the phase and amplitude of the signal.
3. The accuracy in determining the (model) signal by the collocation method depends on the placing of the supporting points x, upon the observed interval and on the coefficient α.
4. The considerations mentioned indicate the possibility of analysing more complicated functions.

Table 1 Signal and correlation function

Signal functions f(x)	Correlation functions K_{ij}	Graph
$\sin x$	$\cos d_{ij}$	Fig. 1
$\sin Bx$	$\cos B d_{ij}$	Fig. 2, $B = 2$
$A \sin (Bx + C)$	$\cos B d_{ij}$	Fig. 3 $A = 0{,}1, B = 2, C = 0$ Fig. 4 $A = 1, B = 2, C = 0.6$
$\exp(-ax) \sin Bx$	$\exp(-a d_{ij} . (\cos Bd_{ij} +$ $+ \sin Bd_{ij} . R(a, B)^{*})$	Fig. 5 $a = 0.5, B = 2.0$
$[A_1 \cos (x + C) \pm$ $\pm A_2 \sin (x + C)] . \sin (Bx)$	$\cos d_{ij} . \cos (B d_{ij})$	Fig. 6 $A_1 = 1, A_2 = 2,$ $B = 2, C = 0.7$
$A \sin (x + C)] . \sin (Bx)$	$\cos d_{ij} . \cos (B d_{ij})$	Fig. 6 $A = 1, B = 2, C = 0.7$

$$^{*})\ R(a,B) = \frac{\dfrac{1}{(2a^2 + 2B^2)}[1 - \exp(-4\pi a) . (a . \sin(4\pi B) + \cos(4\pi B))]}{\dfrac{1}{(4a^2 + B^2)}[\exp(-4\pi a) . (2a . \cos(2\pi B) - B . \sin(2\pi B)) - 2a] + \dfrac{1}{2a}(1 - \exp(4\pi a))}$$

Table 2 Collocation method with one signal with analytically determined correlation function

Signal function f(x)	Trend	Correlation function K_{ij}	α[1])	Δs
$\sin x$	0	$\cos d_{ij}$	$1 . 10^{-6}$	$1 . 10^{-7}$
A[2]) $\sin x$	constant	$\cos d_{ij}$	$1 . 10^{-6}$	$7 . 10^{-6}$
A[3]) $\sin Bx$	0	$\cos 2d_{ij}$	$1 . 10^{-3}$	$5 . 10^{-5}$
A[4]) $\sin x$	constant	$R(d_{ij})$[5]) $\cos d_{ij}$	$1 . 10^{-6}$	$2 . 10^{-6}$

[1]) see the equation (6)
[2]) $A = 10$, trend $= 1000$
[3]) $A = 1/2$, $B = 2$
[4]) $A(x) = 1 - 0{,}1\, x$, trend $= 1000$

[5]) $$R(d_{ij}) = -\frac{0{,}045\,\pi + 0{,}003\,\pi^2}{1 - 0{,}025\,\pi + 0{,}0016\,\pi^2} + \frac{0{,}0025\,\pi - 0{,}1}{1 - 0{,}025\,\pi + 0{,}0016\,\pi^2}\, d_{ij}$$

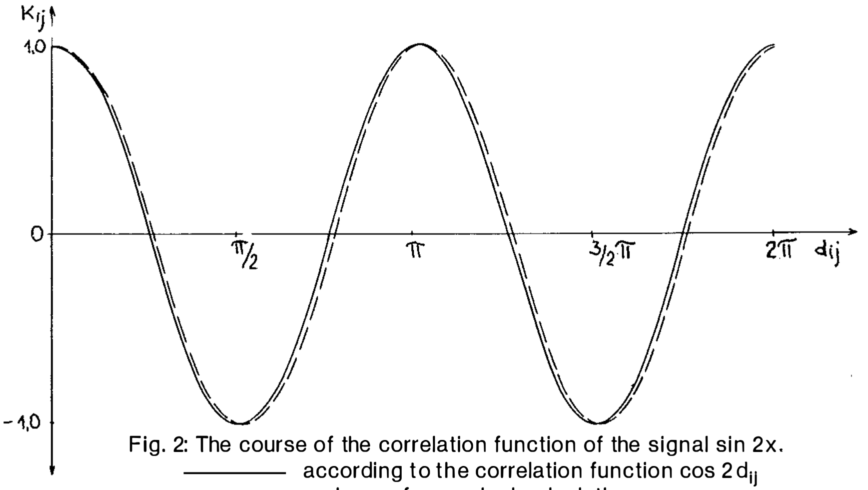

Fig. 2: The course of the correlation function of the signal sin 2x.
——— according to the correlation function $\cos 2d_{ij}$
– – – – on base of numerical calculation

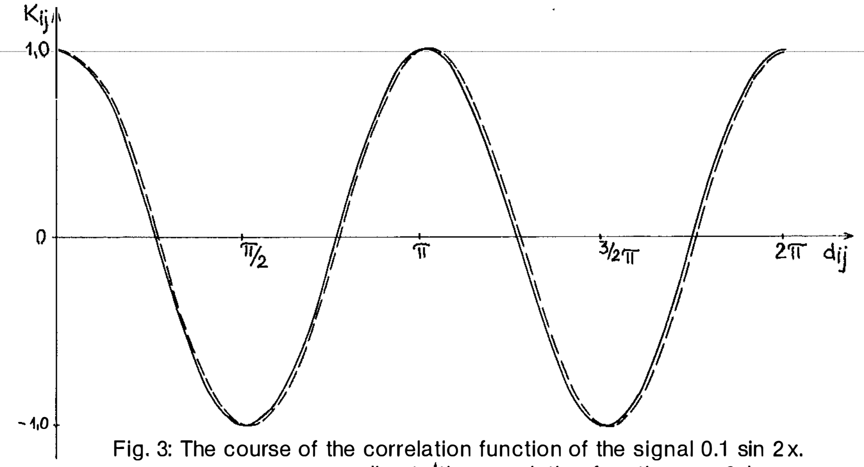

Fig. 3: The course of the correlation function of the signal 0.1 sin 2x.
——— according to the correlation function $\cos 2d_{ij}$
– – – – on base of numerical calculation

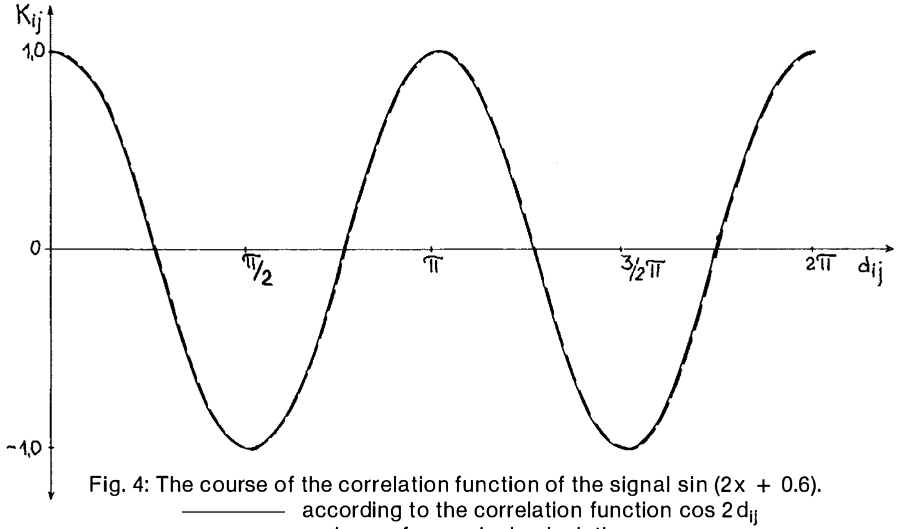

Fig. 4: The course of the correlation function of the signal sin (2x + 0.6).
——— according to the correlation function $\cos 2d_{ij}$
– – – – on base of numerical calculation

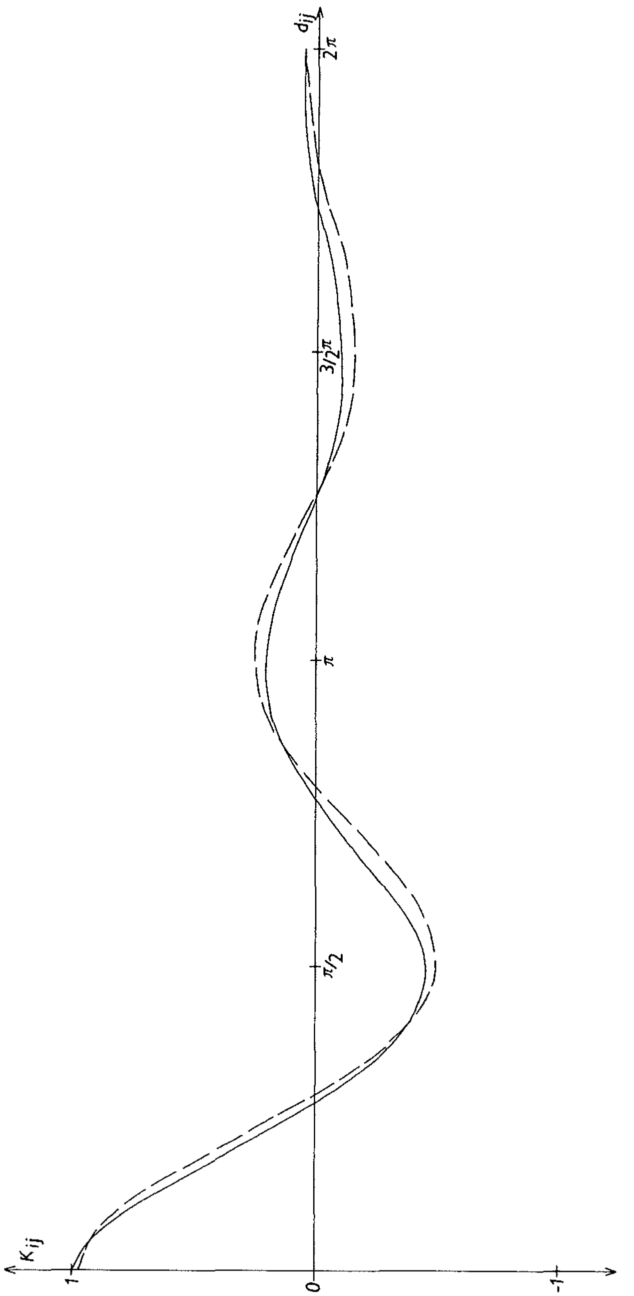

Fig. 5: The course of the correlation function of the signal exp (—1/2x) sin 2x
———— according to the correlation function in the Table 1
— — — — on base of numerical calculation

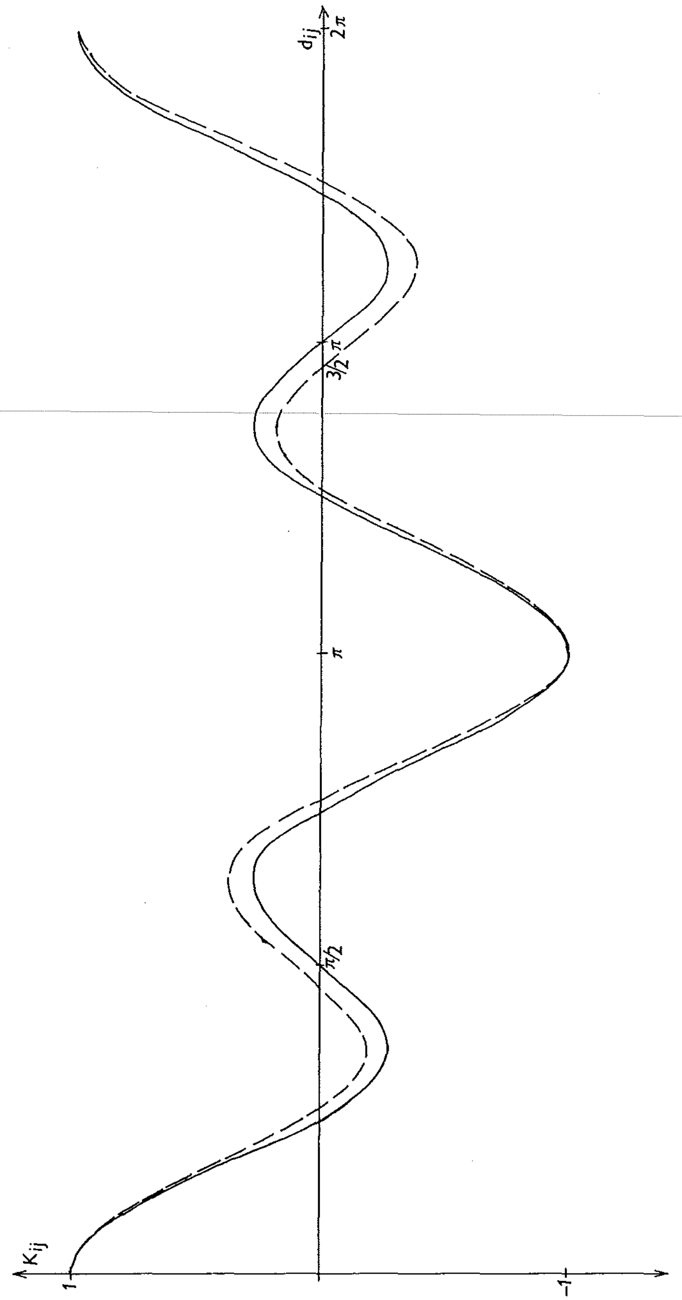

Fig. 6: The course of the correlation function of the signals [cos (x + 0.7) ± 2 sin (x + 0.7)] sin 2x and sin (x + 0.7) sin 2x

——— according to the correlation function cos d_{ij} cos Bd_{ij}

– – – – on base of numerical calculation

3. Collocation with several signals, analytical determination of correlation and cross correlation functions and applications

The fundamental equation of the *fixed* collocation method with *several* signals is

$$\mathbf{A\,x} + \mathbf{s}^1 + \ldots + \mathbf{s}^\sigma - \mathbf{l} - \mathbf{v} = 0, \tag{9}$$

If there are satisfactorily accurate measurements available, i.e. the measurements of such a quality that the noise is lesser than the amplitude of the smallest signal wave and further if all the necessary covariance matrices are known, than it is possible to solve the presented problem. There were derived formulae that are similar to the equation (2) and they are as follows:

$$\mathbf{x} = [\mathbf{A}^T(\mathbf{Q}_l + \sum_{I=1}^{\sigma}\sum_{J=1}^{\sigma}\mathbf{Q}_{s^Is^J})^{-1}\mathbf{A}]^{-1}\,\mathbf{A}^T(\mathbf{Q}_l + \sum_{I=1}^{\sigma}\sum_{J=1}^{\sigma}\mathbf{Q}_{s^Is^J})^{-1}\,\mathbf{l}, \tag{11}$$

$$\mathbf{K} \quad (\mathbf{Q}_l + \sum_{I=1}^{\sigma}\sum_{J=1}^{\sigma}\mathbf{Q}_{s^Is^J})^{-1}\,(\mathbf{l} - \mathbf{A\,x}),$$

$$\mathbf{v} = -\,\mathbf{Q}_l\,\mathbf{K},$$

$$\mathbf{s}^I = \sum_{J=1}^{\sigma}\mathbf{Q}_{s^Is^J}\,\mathbf{K}, \text{ for } I = 1, \ldots \sigma$$

$$\mathbf{s}_p^I = \sum_{J=1}^{\sigma}\mathbf{Q}_{s^Is^J}\,\mathbf{K}, \text{ for } I = 1, \ldots \sigma.$$

In case of a *free* collocation with *several* signals there accedes into the minimalization even the relation $\mathbf{x}^T\mathbf{Q}_{xx}\mathbf{x}$, so that

$$\mathbf{K} = (\mathbf{A\,Q}_{xx}\,\mathbf{A}^T + \mathbf{Q}_l + \sum_{I=1}^{\sigma}\sum_{J=1}^{\sigma}\mathbf{Q}_{s^Is^J})^{-1}\,\mathbf{l}, \tag{12}$$

$$\mathbf{x} = \mathbf{Q}_{xx}\,\mathbf{A}^T\,\mathbf{K}$$

and $\mathbf{v}$, $\mathbf{s}^I$ and $\mathbf{s}_p^I$ see equation (11). Submatrices $\mathbf{Q}_{s^Is^J}$, for $I \neq J$, are cross covariance matrices. If they are zero or if they are supposed to be zero, then

$$\sum_{I=1}^{\sigma}\sum_{J=1}^{\sigma}\mathbf{Q}_{s^Is^J} = \sum_{I=1}^{\sigma}\mathbf{Q}_{s^Is^I}, \tag{13}$$

$$\sum_{J=1}^{\sigma}\mathbf{Q}_{s^Is^J} = \mathbf{Q}_{s^Is^I}, \quad \sum_{J=1}^{\sigma}\mathbf{Q}_{s_p^Is^J} = \mathbf{Q}_{s_p^Is^I}.$$

Estimation of the mean errors is analogous to the paragraph 2.

The approach to the calculation procedures of the equations (11), respectively (12) is possible on base of four different aspects.

The procedure α works with every type of signals $\mathbf{s}^I$, $I = 1, \ldots \sigma$.

The procedure $\alpha 1$ requires to know the particular submatrices $\mathbf{Q}_{s^Is^J}$ for $I, J = I, \ldots, \sigma$, as well as the covariance matrices*) $I = J$, including the cross ones $I \neq J$. By means of one common calculation all the types of signals are determined. Let us designate the procedure $\alpha 1$ as the total collocation.

The procedure $\alpha 2$ requires to know the covariance submatrices $\mathbf{Q}_{s^Is^I}$, $I = 1, \ldots \sigma$. The calculation is carried out by turns.

*) Owing to the form of the equation (11) the concept "covariance" is used. Till next the concept "correlation" matrices, functions and similar will be used.

First with $\mathbf{Q}_{s^1 s^1}$ and the result will be subtracted from the original given values. Then follows the calculation with $\mathbf{Q}_{s^2 s^2}$ etc. There are in total σ calculations. Each of the calculations corresponds to the collocation with one signal. All kinds of signals are obtained without any necessity to know the cross covariances. Let us denote the procedure as a sequential collocation.

The procedure β works with one signal, the equation (10).

$$\mathbf{S} = \sum_{I=1}^{\sigma} \mathbf{s}^I.$$

It requires to know only one covariance matrix $\mathbf{Q}_{ss}$, no cross one, each kind of signals is not determined in particular, but the signal

$$S_i = \sum_{I=1}^{\sigma} s^I_i;$$

where $i = 1, \ldots, n$ and n is the number of measurements.

The procedure $\beta 1$. $\mathbf{Q}_{ss}$ is determined (analytically or numerically) by means of the cumulative signal function. Let us denote it as the collocation of signals sum.

The procedure $\beta 2$. $\mathbf{Q}_{ss}$ is determined by means of a sum of the particular covariance functions that are valid for the particular kinds of signals. Let us denote it as the collocation of covariance functions sum.

It is true in a similar way, with all of the four procedures, about the covariance matrices that are needed for prediction. From the theoretical point of view it is possible to have reservations toward the procedure $\alpha 2$ and first of all to $\beta 2$. From practical point of view to the procedure $\beta 1$.

Analytical determination of correlation and cross correlation functions. The correlation functions K^I_{ij}, $I = 1, \ldots \sigma$, congruent with the equation (5), serve to calculating the elements of correlation submatrices $\mathbf{K}_{s^I s^I}$ that are valid for the signals $\mathbf{s}^I$.

Cross correlation functions K^{IJ}_{ij}, $I, J = 1, \ldots, \sigma$ with $I \neq J$, serve to calculations of the cross correlation submatrix elements $\mathbf{K}_{s^I s^J}$ between the signals $\mathbf{s}^I$ and $\mathbf{s}^J$. It is valid

$$K^{IJ}_{ij} = \frac{\int_0^{x_\pi} (f^I(x)\, f^J(x + d_{ij})\, dx}{\int_0^{x_\pi} f^I(x)\, f^J(x)\, dx}, \quad K^{JI}_{ij} = \frac{\int_0^{x_\pi} (f^J(x)\, f^I(x + d_{ij})\, dx}{\int_0^{x_\pi} f^I(x)\, f^J(x)\, dx}, \tag{14}$$

Similarly to the equation (6) we introduce

$$\mathbf{K}_n = \alpha\, \mathbf{Q}_l + \sum_{I=1}^{\sigma} \sum_{J=1}^{\sigma} \mathbf{K}_{s^I s^I}, \tag{15}$$

see the first and second equation (2), with the possibility that the equation (13) is valid. It is valid in a similar way also for the free collocation, equation (12).

In model application of analytically determined correlation and cross correlation functions on the collocation method with several signals there were used periodical functions again*) in the following form

$$s^I = A_I \sin (B_I\, x + C_I) \tag{16}$$

as partial signals with correlation functions

$$K_{ij} = \cos (B_I\, d_{ij}), \tag{17}$$

*) Particulars in the paragraph 2.

where $d_{ij} = x_j - x_i$, for $l = 1, \ldots, \sigma$. The cross correlation functions were determined analytically from the equation (14) and gave almost null values. But numerical proving has confirmed the results presented only in parts. The values in test were smaller than correlation functions but not zero. Further the cross correlation submatrices were set equal to zero. A simplified model, see the equation (11) has contained:

A polynom of the 2nd degree maximum, $\mathbf{v} = \mathbf{0}$, $s^l = A_l \sin (B_l x + C_l)$, $l = 1, 2$ or $1, \ldots, 3$, $\mathbf{Q}_l = \mathbf{E}$. On the interval $<0,2\pi>$ there were chosen again 18 supporting points ununiformly placed and 10 points to be predicted. To judge the real accuracy ΔS the following relation was chosen

$$\Delta s = \frac{1}{n} \sum_{i=1}^{n} | s_{exact\,i} - s_{colloc\,i} |,$$

where $s_{exact\,i}$ and $s_{colloc\,i}$ are the sums of σ signals that are accurate and calculated by the collocation method with the presented correlation functions, see the Table 3. It follows from the Table:

1. In knowing the correlation functions (17) that depend only on the frequency of the signals, it is possible to solve in total the collocation problem with several signals, equation (16), with required accuracy, when the coefficient α has been chosen suitably.

2. The collocation method fails, when both the frequency and the amplitude are almost the same.

3. The resulting accuracy is practically the same for different calculation procedures, i.e. for procedures $\alpha 1$, $\alpha 2$, $\beta 1$, $\beta 2$, see the paragraph 3. Thus the influence of the problematic cross correlation submatrices is decreased substantially.

Further there was investigated the dependance $\Delta s = \Delta s \left(\alpha, P = \frac{\text{trend}}{\text{amplitude}}\right)$ by means of the function

$$f(x) = 1000 + A \sin x. \tag{18}$$

The graph on the Fig. 7 shows the results. It follows from it:

4. The choise of the coefficient α, if the ratio P is known and if the required resulting accuracy is Δs.

By means of the relation

$$f(x) = 1000 + A (\sin B_1 x + \sin B_2 x) \tag{19}$$

there was investigated further the dependance $\Delta S = \Delta S (\Delta B = B_2 - B_1, P)$.
The graph on the Fig. 8 shows the result. It follows from the graph:

5. In knowing the ratio P and the difference ΔB of the frequencies it is possible to do a priori the estimation of the resulting accuracy ΔS.

The third paragraph can be concluded with stating that in knowing the correlation functions K^l_{ij}, where $l = 1, \ldots, \sigma$ and σ is the number of the particular kinds of signals, it is possible to do the calculation by the collocation method with the required accuracy including the calculation of the particular predicted signals, trend and values. The correlation functions are dependent only on signal function frequencies. The influence of the cross correlation submatrices is not substantial. The signal functions of the given equation (7) were taken into account here.

4. Anharmonic analysis by the collocation method

In the preceding paragraphs 2 and 3, there have been derived correlation functions that contain only the frequencies of partial signal functions (waves), see the equation

Table 3 Collocation method with several signals with analytically determined correlation functions

Signal function f(x)	Trend	Procedure of the calculation*)	Function: correlation	Function: cross correlation	α	ΔS	Note
$1.0 \sin x + 0.5 \sin 2x$	linear	collocation of signals sum — $\beta 1$	$0.8 \cos_{ij} + 0.2 \cos 2 d_{ij}$	no	10^{-3}	10^{-4}	Accuracy congruent
	linear	collocation of correlation functions sum — $\beta 2$	$\cos d_{ij} + \cos 2 d_{ij}$	no	10^{-3}	10^{-4}	
$0.1 \sin x + 0.125 \sin (0.9x + 0.6)$	quadratic	collocation of correlation functions sum — $\beta 2$	$\cos d_{ij} + \cos 0.9 d_{ij}$	no	10^{-5}	2.10^{-3}	Accuracy decreased for proximity of both functions
$0.1 \sin x + 0.125 \sin (0.99x + 0.6)$	quadratic	collocation of correlation functions sum — $\beta 2$	$\cos d_{ij} + \cos 0.99 d_{ij}$	no	10^{-5}	10^{-2}	
$1.0 \sin x + 0.1 \sin (x + 0.6)$	linear	sequential collocation — $\alpha 2$	I. $\cos d_{ij}$ II. $\cos d_{ij}$	no	10^{-4}	10^{-5} 10^{-6}	Accuracy congruent both in particular signals
	linear	collocation of correlation functions sum — $\beta 2$	$\cos d_{ij} + \cos d_{ij}$	no	10^{-4}	10^{-5} 10^{-6}	
$1.0 \sin x + \sin (1.2x + 0.6) + 0.125 \sin 4x$	linear	collocation of signals sum — $\beta 1$	$\cos d_{ij} + \cos 1.2 d_{ij} + \cos 4 d_{ij}$	no	10^{-5}	10^{-5} 10^{-5} 10^{-6}	Accuracies of particular signals

*) Paragraph 3.

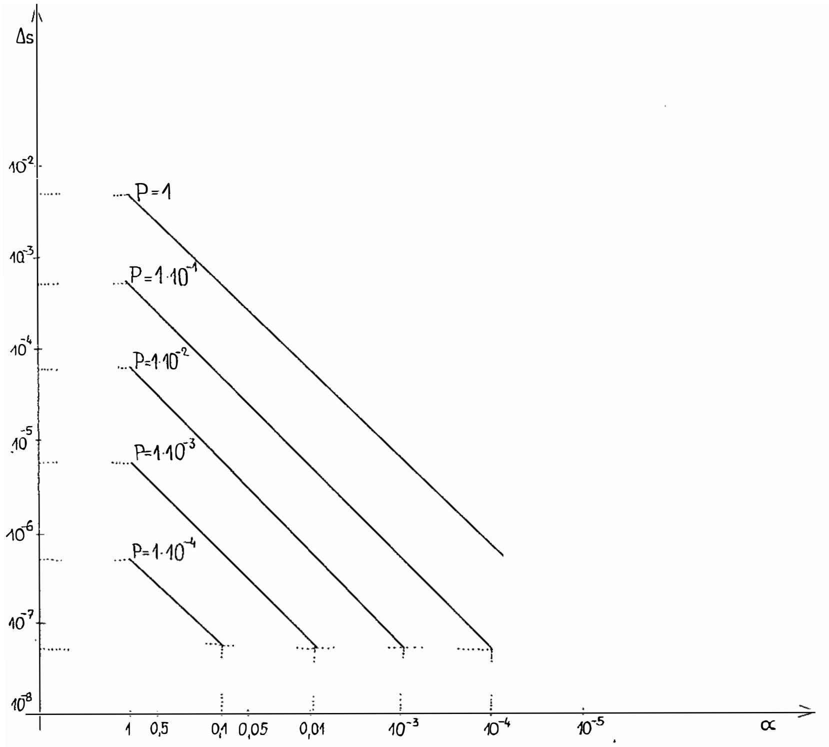

Fig. 7: Dependance of the accuracy Δs — in the equation (16) I = 1 — on the coefficient α and the ratio P = amplitude/trend; for the equation (18)

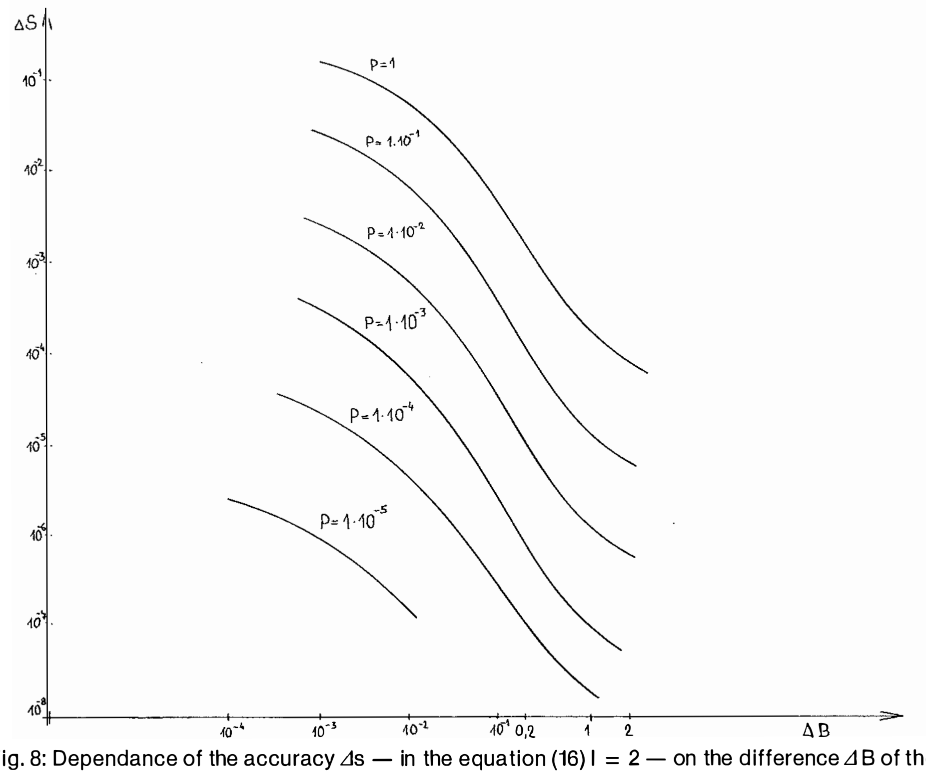

Fig. 8: Dependance of the accuracy Δs — in the equation (16) I = 2 — on the difference ΔB of the frequencies B_1, B_2 and the ratio P = amplitude/trend; for the equation (19).

(17). The amplitude and the phase shift are dropped. There was further proved numerically that the correlation functions manage to predict various partial sinusoidal*) types of signals. From the viewpoint of the discussed problems even the inverse task may have its sense. It reads: a set of supporting values is given by discrete values and is influenced by various (partial) types of sinusoidal signals. Their frequencies are to be found, i.e. the total signal should be spread to partial sinusoids. As indicator works the expression $\mathbf{v}^T\mathbf{v} = f(B_1, \ldots B_\sigma)$, where B_l is the frequency of l signal type, $l = 1, \ldots, \sigma$ and σ is the number of signal types. If it is true

$$\mathbf{v}^T\mathbf{v} = f(B_1, \ldots, B_\sigma) = \min, \tag{20}$$

then $B_1, \ldots, B\sigma$ are the frequencies of the partial sinusoidal signals that are to be found.

4.1 Anharmonic analysis by the collocation method for one signal

Let the signal have the following form:

$$s = A \sin (Bx + C), \tag{21}$$

see the equation (7), where A is the amplitude, C is the phase shift, $B = \frac{2\pi}{T}$ is the frequency and T is the period. The correlation function of signal (21) is

$$K_{ij} = \cos Bd_{ij} \tag{22}$$

see the equation (8). The analysis is done again under presumption that the properties of a random quantity are adjudged to the deterministic component sin (Bx + C) and accordingly the least square method is appplied.**)

The graph on the Fig. 9 shows the dependence $\mathbf{v}^T\mathbf{v} = f(B)$ of the signal function $s = \sin x$, accordingly for $B = 1.0$. On the interval $<0,2\pi>$ there were chosen again 18 supporting values, linear trend and $\mathbf{v} = \mathbf{0}$. The equation (6) for $\alpha = 10^{-6}$ was used, $Q_l = E$ and the equation (22). It follows from the course of the curve, that the minimum corresponds to the given frequency.

The frequency B, which is to be found out, can be defined with more precision for example by means of the relation

$$\frac{(B_i - B)^2}{2\,p} = (\mathbf{v}^T\mathbf{v})_i - \mathbf{v}^T\mathbf{v},\ i = 1, \ldots, n \tag{23}$$

that substitutes the function $\mathbf{v}^T\mathbf{v} = f(B)$ in the surrounding of the minimum by the quadratic parabola. Particulars see Appendix 1. Defining the frequency with more precision (Appendix 1) is suitable, if only a rough estimation of approximate frequency is known. Thus the convergence can be speeded up. With the more precise frequency the detailed calculation is then done by the collocation method in surrounding of the extreme; the development $\mathbf{v}^T\mathbf{v}$ is followed and the frequency is defined with more precision again— see Appendix 1.

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*) Included damped sinusoids.

**) Particulars are in the paragraph 2.

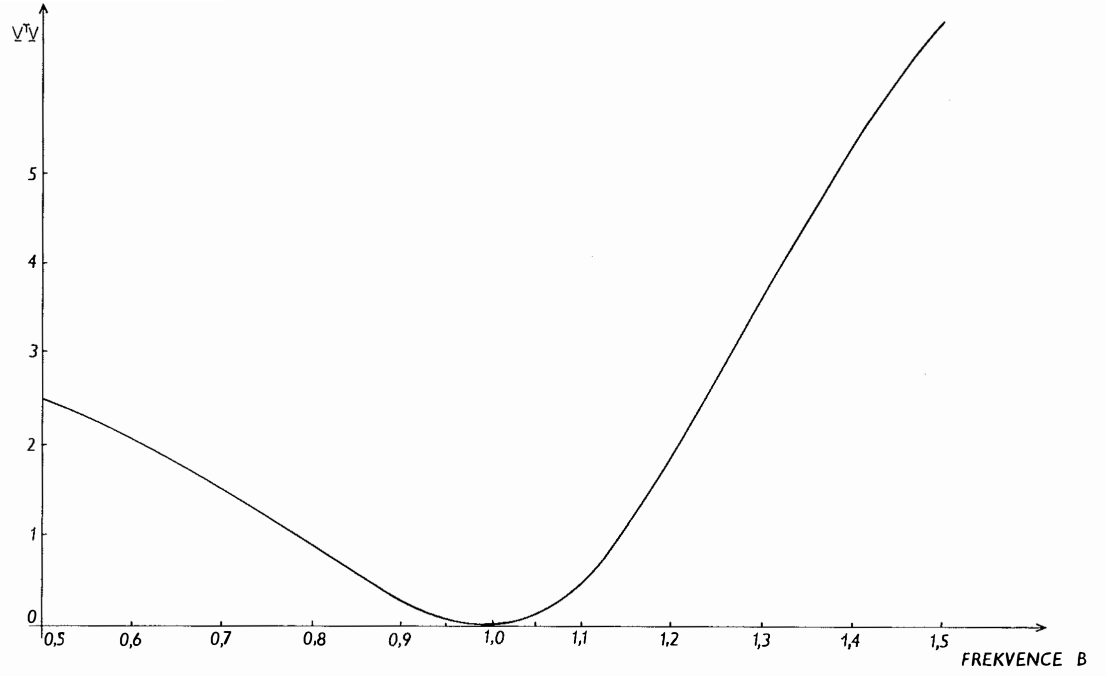

Fig. 9: The course of $\mathbf{v}^T\mathbf{v}$ in the dependance of the frequency B of the correlation function cos B_{ij} for the signal function s = sin x, accordingly for B = 1.0.

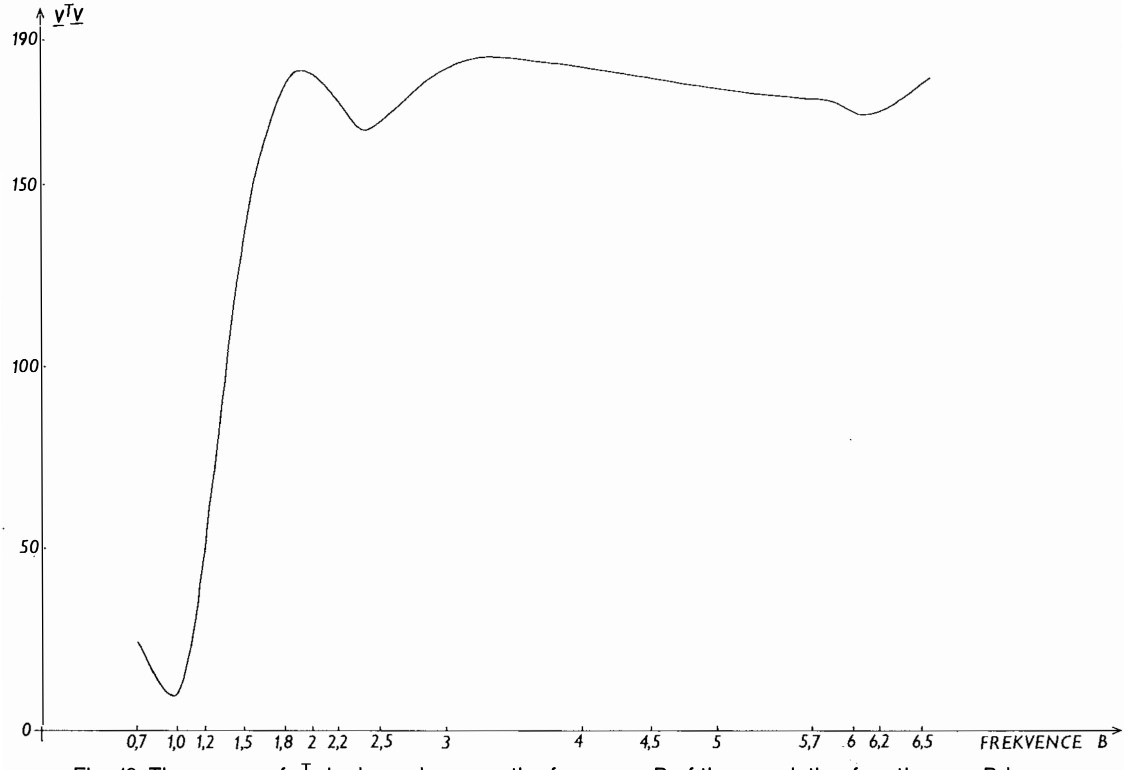

Fig. 10: The course of $\mathbf{v}^T\mathbf{v}$ in dependance on the frequency B of the correlation function cos Bd_{ij} of the signal function s = 5.0 sin x + sin (2.0 x + 0.5) + 0.3 sin (6.0 x + 1), accordingly for B_1 = 1.0, B_2 = 2.0 and B_3 = 6.0.

4.2 Anharmonic analysis by the collocation method for several signals

Let the total signal have the form

$$S = \sum_{l=1}^{n} A_l \sin (B_l x + C_l), \tag{24}$$

where σ is the number of the particular signal types, see the equation (16). To each of them belongs the correlation function

$$K_{lj} = \cos B_l d_{lj},$$

see the equation (17). It is necessary to do the analysis again under the presumption that the properties of random quantities can be adjudged to the deterministic components $\sin (B_l x + C_l)$ and the least square method can be used as well*). So the task is to find the frequencies B_l in the equation (24) by means of the condition (20).

Determination of the approximate values of the frequencies B_l was first proved on numerical examples by means of the procedure $\alpha 1$ in the paragraph 3. Finally there has appeared as most suitable the procedure that followed the values $\mathbf{v}^T\mathbf{v}$ for one correlation function

$$K_{ij} = \cos Bd_{ij}, \tag{25}$$

in which the frequency B was changed, the signal S was taken into and this can be critisized from the theoretical point of view. The graph in the Fig. 10 presents the dependence $\mathbf{v}^T\mathbf{v} = f(B)$ of the signal function

$$s = 5.0 \sin x + \sin (2.0 x + 0.5) + 0.3 \sin (6.0 x + 1.0), \tag{26}$$

accordingly for $B_1 = 1.0$, $B_2 = 2.0$ and $B_3 = 6.0$. On the interval $<0,2\pi>$ there were chosen again 18 supporting values, linear trend and $\mathbf{v} = \mathbf{0}$. The equation (6) was used for $\alpha = 10^{-6}$, $\mathbf{Q}_l = \mathbf{E}$ and the equation (25).

It follows from the course of the curve that the minimums correspond approximately to the given frequencies.

The same procedure was applied also for example with an unzero vector v of the corrections, namely on the set of values that are presented in the Fig. 11. There the course of the mean standard errors m_0 of the evaluated astrometric plates is presented in the dependance on the rectascension $\alpha \in <0,24h>$, (9).

The course of $\mathbf{v}^T\mathbf{v} = f(B)$ with the application of the equation (25) is presented on the graph in Fig. 12. The minimums give the frequences respective the periods that can be physically interpreted.

The procedure using only the equation (25) can be combined with the procedure $\alpha 2$—sequential collocation in the paragraph 3. But here after neglecting the other signals different value of the frequency, that is looked for, can occur. The procedure $\alpha 1$—the total collocation was applied as well. But it is rather complicated.

If the approximative values of the frequencies are known, there can be used a similar procedure as in analysing the single signal to make them more precise. The equation (23) changes into the form

$$\sum_{i=1}^{\sigma} \frac{(B_{li} - B_l)^2}{2\, a_l^2} = (\mathbf{v}^T\mathbf{v})_i - \mathbf{v}^T\mathbf{v},\ i = 1, \ldots, n, \tag{27}$$

that substitutes the function $\mathbf{v}^T\mathbf{v} = f(B_l, \ldots, B_\sigma)$ in the surrounding of the minimum by σ-dimensional paraboloid. Particulars see Appendix 2.

*) Particulars in the paragraph 2.

With more precise frequencies the detailed calculation by the collocation method with several signals in the surrounding of the extremes can be carried out, for example according to the procedure α2-sequential collocation. Again the development $\mathbf{v}^T\mathbf{v}$ is followed and again the frequency is made more precise—see Appendix 2.

Up to now there has not been discussed the trend with the analysis of the signals. This degrading influence can be removed by reducing the measured quantities by approximative trend that is determined by the polynom of suitable degree. In the course of the calculating procedure it is possible to make the trend more precise.

Accuracy estimation. Collocation with one signal gives the formula for the calculation of the mean errors of the unknown trend coefficients, of the predicted signals, of the predicted values and similar. Similarly also the collocation with more signals gives the same for each type in particular when the individual covariance matrices are known. The mean quadratic errors of the frequencies B and B_l result from the least square method in adjusting the systems (28) and (29).

5. Conclusion

The contribution links up with the "classic" collocation method. Several signals and the free collocation are applied. The covariance functions are substituted by correlation ones and filtering by eligible coefficient α, that serves also to tuning the numerical and signal values and to inverting the matrices (6) and (15). It has been proved that the analytically determined correlation functions catch up fully each partial signal and enable their prediction with satisfying accuracy. The cross correlation functions do not have any fundamental importance.

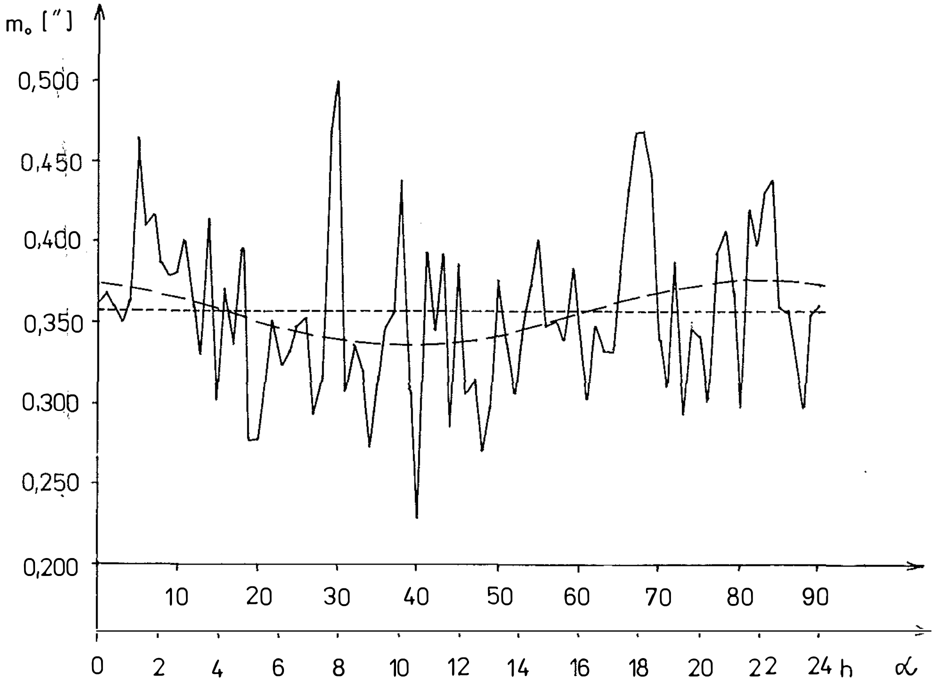

Fig. 11: The course of the mean standard errors m_0 of the astrometric plates in dependance on the rectascension α, (9).

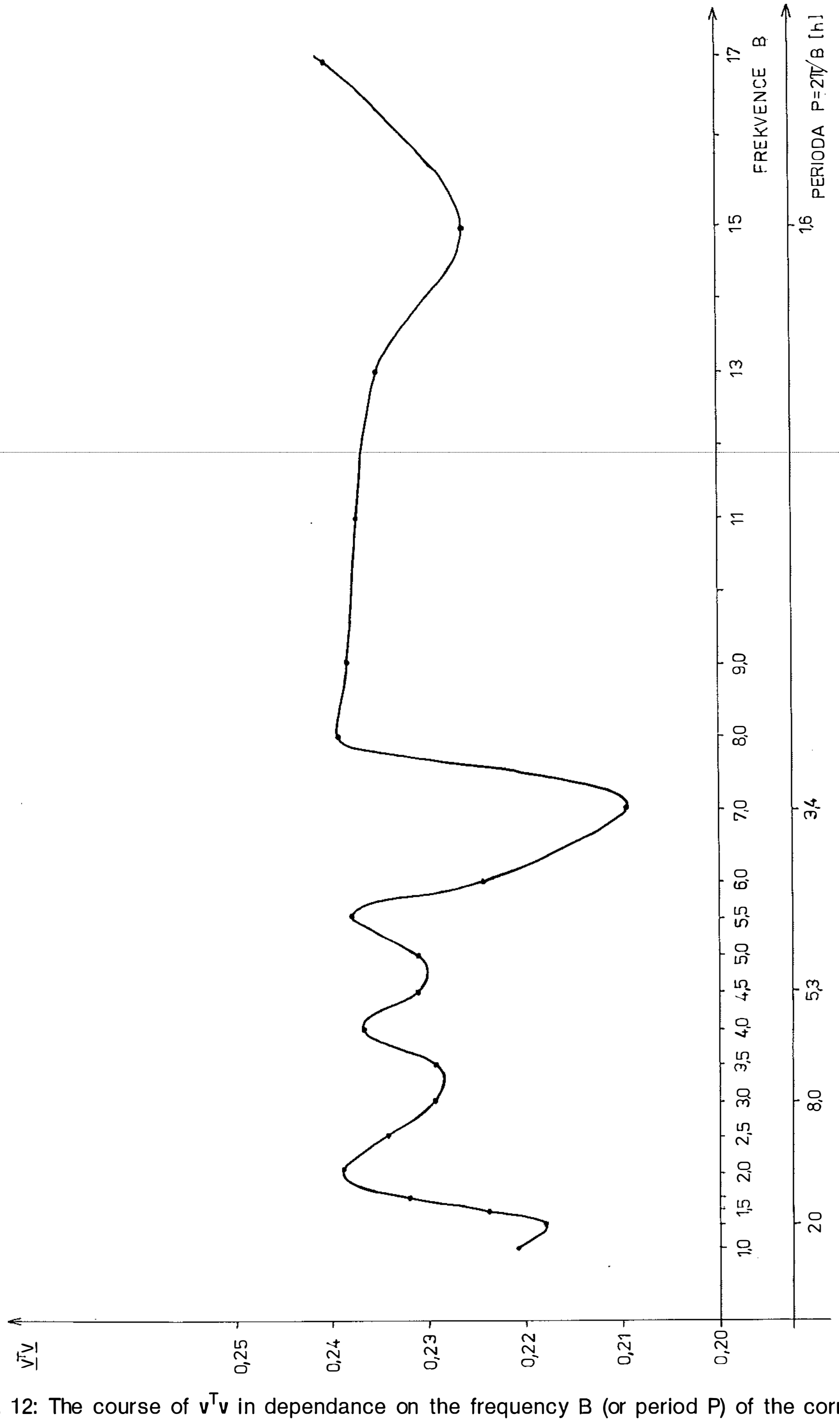

Fig. 12: The course of $v^T v$ in dependance on the frequency B (or period P) of the correlation function cos Bd_{ij} of the discrete values illustrated on the Fig. 11.

Due to the consequential fact that the correlation function includes only the frequency of the sinusoidal signal function, it is possible to use the collocation method to anharmonic analysis of the given discrete values, i.e. to determination of the individual frequencies B_l of the partial signal waves s^l, $l = 1, \ldots, \sigma$ and σ is the number of the signal types. The total signal $S = \sum_{i=1}^{\sigma} s^l$. The criterion is the condition $\mathbf{v}^T\mathbf{v} = \min.$, see the equation (20). The method can be used also for the trend different from zero even if it is substituted by a polynom (the simpliest case) or if it is given commonly with optional discrete values. The trend could be removed by the classic collocation method.

To prove it, there was used a model example with null random dispersion, see the equation (26) and Fig. 10, as well as the example with unzero dispersion, see Fig. 11 and 12.

With respect to the results in the Table 3 in the 3rd paragraph it appears that it will be possible to analyse both the functions with periods being not only mutually near but also near to the interval, upon which the set is being analysed. It is also possible, see the last example in the Table 2 in the 2nd paragraph, to approach the analysis of also such partial signal functions, that have no sinusoidal character. The presented anharmonic analysis does not require an equidistant pace and thus the aliasing hazard is decreased.

Appendix 1

Making more precise the frequency B for one signal that is looked for

It links up with the equation (23). After introducing the auxiliary unknown quantities $p_0 = \mathbf{v}^T\mathbf{v} + B^2/2p$, $p_1 = -B/p$, $p_2 = 1(2p)$, the equation (23) changes into the form

$$p_0 + p_1 B_i + p_2 B_i^2 = (\mathbf{v}^T\mathbf{v})_i, \quad i = 1, \ldots, n. \tag{28}$$

With the necessary number of observations $n = 3$. If $n > 3$, the adjustment occurs, where the mediating equation of corrections is just the equation (28). The frequency with more precision is then $B = -p_1/(2p_2)$.

Appendix 2

Making more precise the frequencies B_l for more signals to be looked for

It links up with the equation (27). After introducing the auxiliary unknown quantities

$$p_0 = \mathbf{v}^T\mathbf{v} + \sum_{i=1}^{\sigma} B_l^2/2\,a_l^2,\ p_{1l} = -B_l/a_l^2,\ p_{2l} = 1/2\,a_l^2,\ l = 1, \ldots, \sigma$$

the equation (27) changes into the form

$$p_0 + \sum_{i=1}^{\sigma} p_{1,l} B_{li} + \sum_{i=2}^{\sigma} p_{2,l} B_{li}^2 = (\mathbf{v}^T\mathbf{v})_i, \quad i = 1, \ldots, n. \tag{29}$$

With a necessary number of observations $n = 2\sigma + 1$. If $n > 2\sigma + 1$, adjustment will be done, where the mediating equation of corrections is just the equation (29). The frequencies with more precision being looked for are $B_l = -p_{1,l}/(2\,p_{2,l})$, $l = 1, \ldots, \sigma$.

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