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# An Efficient Technique for Harmonic Analysis on a Spheroid (Ellipsoid and Sphere)

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# An Efficient Technique for Harmonic Analysis on a Spheroid (Ellipsoid and Sphere)

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**Abstract**

The paper presents an efficient technique for harmonic analysis on a spheroid (both sphere and ellipsoid). Colombo (1981) has introduced an effective and fast technique for the harmonic analysis on the sphere using Fast Fourier Technique FFT. It is known that there is no direct mathematical relationship to transform the point or/and mean gravitational observables from the surface of the mean earth's ellipsoid to the surface of the sphere. Hence, the paper introduces an efficient program HRCOFITR that uses essentially Colombo's main subroutines HARMIN and SSYNTH, after significant and critical modifications written by the author, in an iterative and scaling process for harmonic analysis on both the sphere and the ellipsoid. In order to check the performance of the proposed technique, two computational tests have been carried out. In the first test, two data fields on the ellipsoid have been created using OSU91A geopotential model. In the second test a data field on the sphere has been created using OSU91A geopotential model. In all cases, the harmonic coefficients have been analyzed using Colombo's technique as well as using the developed technique (HRCOFITR program). The results proved that the developed technique gives better accuracy for the estimated harmonic coefficients as well as for the residual field in all cases.

**Zusammenfassung**

Die vorliegende Arbeit präsentiert eine leistungsfähige Methode zur harmonischen Analyse auf einem Sphäroid (Kugel wie auch Ellipsoid). Colombo (1981) hat erstmals einen effektiven und schnellen Algorithmus für die harmonische Analyse auf der Kugel unter der Verwendung der Fast Fourier Technik (FFT) vorgestellt. Bekannterweise existiert kein direkter mathematischer Zusammenhang, um Punkt und/oder Mittelwerte der Schwere von der Oberfläche eines mittleren Erdellipsoids auf die Oberfläche einer Kugel zu transformieren. Aus diesem Grund wird in dieser Arbeit ein leistungsfähiges Programm HRCOFITR, das wesentlich auf den beiden zentralen Unterprogrammen HARMIN und SSYNTH von Colombo aufbaut, vorgestellt. Die beiden Unterprogramme wurden kritisch untersucht und signifikant vom Autor modifiziert und in einen iterativen Skalierungsprozess zur harmonischen Analyse auf der Kugel und dem Ellipsoid verwendet. Um die Leistungsfähigkeit der verwendeten Methode zu zeigen, wurden zwei Tests durchgeführt. In einem ersten Test wurden zwei Datenfelder auf dem Ellipsoid unter Verwendung des Kugelfunktionsmodells OSU91A erstellt. In einem zweiten Test wurden ein Datensatz auf der Kugel mittels des Kugelfunktionsmodells OSU91A erstellt. In allen Fällen wurden die harmonischen Koeffizienten sowohl mit Hilfe von Colombos Technik, wie auch der neu entwickelten Technik (HRCOFITR Programm) analysiert. Die Ergebnisse zeigen das die neu entwickelte Methode sowohl für den Fall der geschätzten harmonischen Koeffizienten, wie auch für den Fall des Restfeldes, in allen Fällen eine bessere Genauigkeit liefert.

## 1. Introduction

A huge amount of global gravitational data became recently available. This has enhanced the resolution of the developed geopotential earth models. Basically, each gravitational observable gives a normal equation in terms of the unknown geopotential coefficients. Thus, we face a very huge system of normal equations, which needs a special technique of solving such a terrible Least-Squares problem using the relatively limited computer facilities. Colombo (1981) has introduced a very powerful method for harmonic analysis on the sphere using Fast Fourier Technique FFT.

Usually the gravitational data are collected on the surface of the earth and then reduced to the surface of the mean earth's ellipsoid (representing the only accepted spheroid to regenerate the potential of the earth), see, e.g., (Vanicek and Krakiwsky, 1982). Unfortunately, no direct mathematical relationship exists to transform the point or/and mean gravitational observables from the surface of the mean earth's ellipsoid to the surface of the sphere. Hence, direct application of Colombo's FFT technique in practice seems to be impossible.

It should be noted that there exists some work done to transform the spherical to ellipsoidal harmonics (and vice versa), but not transforming the point or/and mean gravitational observable from the surface of the mean earth's ellipsoid to the surface of the sphere; see, e.g., (Petrovskaya and Verslikov, 2000; Petrovskaya et al., 2001; Blais and Provins, 2002; Grafarend et al., 1999).

The paper presents a technique that uses Colombo's main subroutines HARMIN and

SSYNTH, after significant and critical modifications implemented by the author, in an iterative and scaling process for the harmonic analysis of data on the surface of both the sphere and the ellipsoid. For the sake of checking the developed technique, two computational tests have been carried out. In the first test, two data fields on the ellipsoid have been created using the OSU91A geopotential model. In the second test a data field on the sphere has been created using the OSU91A geopotential model. In all cases, the harmonic coefficients have been analyzed using Colombo's technique as well as using the developed technique. A comparison between the computed coefficients and the OSU91A coefficients along with the residual field in each case has been made and widely discussed.

It should be noted that there exist some work done in the field of efficient computational techniques of spherical harmonics; see, e.g., (Driscoll and Healy, 1994; Mohlenkamp, 1999; Petrovskaya et al., 2001).

## 2. Spherical Harmonic Analysis

Let us consider an analytical function $f(\theta,\ \lambda)$ defined on the unit sphere ($0 \leq \theta \leq \pi$ and $0 \leq \lambda \leq 2\pi$). Expand $f(\theta,\ \lambda)$ in series of surface spherical harmonics (Moritz, 1980, p. 21)

$$f(\theta,\ \lambda) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (\overline{C}_{nm} \cos m\,\lambda + + \overline{S}_{nm} \sin m\,\lambda)\, \overline{P}_{nm}(\cos\theta), \tag{1}$$

where $\overline{C}_{nm}$and $\overline{S}_{nm}$ are the fully normalized spherical harmonic coefficients and refers to the fully normalized associated Legendre functions. Let us introduce the abbreviations

$$\begin{aligned} \overline{R}_{nm}(\theta,\lambda) &= \overline{P}_{nm}(\cos\theta)\cos m\,\lambda, \\ \overline{Q}_{nm}(\theta,\lambda) &= \overline{P}_{nm}(\cos\theta)\sin m\,\lambda. \end{aligned} \tag{2}$$

It is well known that the fully $\overline{P}_{nm}(\cos\theta)$ normalized harmonic coefficients are orthogonal, i.e., they satisfy the orthogonality relations (Heiskanen and Moritz, 1967, p. 29–31)

$$\begin{aligned} &\iint_\sigma \overline{R}_{nm}(\theta,\lambda)\overline{R}_{n'm'}(\theta,\lambda)\, d\sigma = \\ &\iint_\sigma \overline{Q}_{nm}(\theta,\lambda)\overline{Q}_{n'm'}(\theta,\lambda)\, d\sigma = 0, \end{aligned} \tag{3}$$

$$\iint_\sigma \overline{R}_{nm}(\theta,\lambda)\overline{Q}_{nm}(\theta,\lambda)\, d\sigma = 0, \tag{4}$$

$$\frac{1}{4\pi}\iint_\sigma \overline{R}^2_{nm}(\theta,\lambda) = \frac{1}{4\pi}\iint_\sigma \overline{Q}^2_{nm}(\theta,\lambda) = 1, \tag{5}$$

As a consequence of the orthogonality, the fully normalized harmonic coefficients $\overline{C}_{nm}$ and $\overline{S}_{nm}$ can be given by (ibid, p. 31)

$$\begin{aligned} \overline{C}_{nm} &= \frac{1}{4\pi}\iint_\sigma f(\theta,\lambda)\,\overline{R}_{nm}(\theta,\lambda)\, d\sigma, \\ \overline{S}_{nm} &= \frac{1}{4\pi}\iint_\sigma f(\theta,\lambda)\,\overline{Q}_{nm}(\theta,\lambda)\, d\sigma. \end{aligned} \tag{6}$$

In fact, (6) cannot be used in practice to compute the harmonic coefficients $\overline{C}_{nm}$and $\overline{S}_{nm}$ simply because the analytical function $f(\theta,\lambda)$ is generally unavailable. Only a finite set of *noisy* measurements $f(\theta_i,\lambda_j)$, covering the whole sphere, might be available. Discretizing (6) on an equal angular grid covering the whole sphere gives the following quadrature formulas

$$\begin{aligned} \hat{C}_{nm} &= \frac{1}{4\pi}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} f(\theta_i,\lambda_j)\,\overline{R}_{nm}(\theta_i,\lambda_j)\,\Delta_{ij}, \\ \hat{S}_{nm} &= \frac{1}{4\pi}\sum_{i=0}^{N-1}\sum_{j=0}^{N-1} f(\theta_i,\lambda_j)\,\overline{Q}_{nm}(\theta_i,\lambda_j)\Delta_{ij}, \end{aligned} \tag{7}$$

where $\hat{C}_{nm}$and $\hat{S}_{nm}$ are the estimate of $\overline{C}_{nm}$and $\overline{S}_{nm}$, respectively, $\Delta_{ij}$ indicates the segment area and $N$ is the number of grids in the latitude direction. Expression (7) is used to compute the harmonic coefficients if the available data field is represented by a set of point values $f(\theta_i,\lambda_j)$. It should be noted that (7) is usually only an *approximation* due to the discretization effect of $f(\theta,\lambda)$.

The data field could be represented by area mean values $\overline{f}(\theta_i,\lambda_j)$. Expanding the area means $\overline{f}(\theta_i,\lambda_j)$ is done by integrating (1), term-by-term, which gives

$$\begin{aligned} \overline{f}(\theta_i,\lambda_j) &= \frac{1}{\Delta_{ij}}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\iint_{\sigma_{ij}} (\overline{C}_{nm}\cos m\,\lambda + \\ &\quad +\overline{S}_{nm}\sin m\,\lambda)\,\overline{P}_{nm}(\cos\theta)\, d\sigma \\ &= \frac{1}{\Delta_{ij}}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\int_{\theta_i}^{\theta_i+\Delta\theta}\overline{P}_{nm}(\cos\theta)\sin\theta\, d\theta\,\cdot \\ &\quad \cdot\left(\overline{C}_{nm}\int_{\lambda_j}^{\lambda_j+\Delta\lambda}\cos m\,\lambda\, d\,\lambda + \overline{S}_{nm}\int_{\lambda_j}^{\lambda_j+\Delta\lambda}\sin m\,\lambda\, d\lambda\right) \end{aligned} \tag{8}$$

where $\overline{f}(\theta_i, \lambda_j)$ is the area mean of $f(\theta, \lambda)$ on the block $\sigma_{ij}$ whose area is $\Delta_{ij}$ given by

$$\Delta_{ij} = \Delta\lambda[\cos\theta_i - \cos(\theta_i + \Delta\,\theta)]. \qquad (9)$$

If the data field is represented by area means $\overline{f}(\theta_i, \lambda_j)$, expression analogous to (7) is proposed (Colombo, 1981, p. 4)

$$\hat{C}_{nm} = \mu_n \sum_{i=0}^{N-1} \sum_{j=0}^{2N-1} \overline{f}(\theta_i, \lambda_j) \int\limits_{\theta_i}^{\theta_i+\Delta\theta} \overline{P}_{nm}(\cos\,\theta) \sin\theta\, d\,\theta \int\limits_{\lambda_j}^{\lambda_j+\Delta\lambda} \cos m\,\lambda\, d\lambda, \qquad (10)$$

$$\hat{S}_{nm} = \mu_n \sum_{i=0}^{N-1} \sum_{j=0}^{2N-1} \overline{f}(\theta_i, \lambda_j) \int\limits_{\theta_i}^{\theta_i+\Delta\theta} \overline{P}_{nm}(\cos\,\theta) \sin\theta\, d\,\theta \int\limits_{\lambda_j}^{\lambda_j+\Delta\lambda} \sin m\,\lambda\, d\lambda,$$

where $\mu_n$ denotes the de-smoothing factor. Colombo (1981) has used the following definition of the de-smoothing factor

$$\mu_n = \frac{1}{4\,\pi\,\eta_n} \qquad (11)$$

with

$$\eta_n = \begin{cases} \beta_n^2 & \text{if } 0 \le n \le \frac{1}{3N} \\ \beta_n & \text{if } \frac{1}{3N} \le n \le N \\ 1 & \text{if } n > N \end{cases} \qquad (12)$$

where $\beta_n$ is known as the Pellinen smoothing factor of degree $n$ given by (Meissl, 1971, p. 24)

$$\beta_n = \frac{1}{1-\cos\psi_\circ}\frac{1}{2n+1}\left[P_{n-1}(\cos\psi_\circ) - P_{n+1}(\cos\psi_\circ)\right] \qquad (13)$$

with (Colombo, 1981, p. 85)

$$\cos\psi_\circ = \frac{\Delta\lambda}{2\pi}(cos\,\theta_{i+1} - \cos\,\theta_i) + 1, \qquad (14)$$

and $P_n(\cos\,\theta)$ is the Legendre polynomial. $N$ appearing in (12) is the so-called Nyquist frequency (see below).

If all $\overline{C}_{nm}$ and $\overline{S}_{nm}$ are known till degree and order $N_{max}$, one can compute $\tilde{f}(\theta_i, \lambda_j)$ and $\tilde{\overline{f}}(\theta_i, \lambda_j)$ as follows:

$$\tilde{f}(\theta_i, \lambda_j) = \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} (\overline{C}_{nm}\cos m\lambda_j + \overline{S}_{nm}\sin m\lambda_j)\,\overline{P}_{nm}(\cos\theta_i) \qquad (15)$$

$$\tilde{\overline{f}}(\theta_i, \lambda_j) = \frac{1}{\Delta_{ij}} \sum_{n=0}^{N_{max}} \sum_{m=0}^{n} \int\limits_{\theta_i}^{\theta_1+\Delta\theta} \overline{P}_{nm}(\cos\theta)\sin\theta\; d\theta \left(\overline{C}_{nm}\int\limits_{\lambda_j}^{\lambda_j+\Delta\lambda}\cos m\lambda\, d\lambda + \overline{S}_{nm}\int\limits_{\lambda_j}^{\lambda_j+\Delta\lambda}\sin m\lambda\, d\lambda\right) \qquad (16)$$

which can be regarded as an *approximation* to $f(\theta, \lambda)$ and $\overline{f}(\theta, \lambda)$, respectively, at point $(\theta_i, \lambda_j)$. Expressions (15) and (16) define the object of spherical harmonic synthesis: given the coefficients, it is required to estimate the function.

The double summation appearing in (7) and (10) for harmonic analysis as well as in (15) and (16) for spherical harmonic synthesis are computed using FFT technique. Colombo (1981) has written two subroutines for this purpose, called HARMIN and SSYNTH.

It should be noted that there are two conditions, which should be satisfied to use FFT in harmonic analysis and synthesis. They are:

- The number of grid blocks of the data field should "stride" the equator, i.e., the grid should be symmetric with respect to the equator. In other words, $N$ should always be even.
- The maximum recoverable degree $N_{max}$ should be smaller than the Nyquist frequency $N$ defined by

$$N = \frac{\pi}{\Delta\lambda}. \qquad (17)$$

This follows Nyquist theorem stating: the Fourier coefficients of a function of period $2N\,\Delta\lambda$ can be recovered only if $N_{max} < N$ (Elliott and Rao, 1982, p. 38).

## 3. Spherical Harmonics Expansions of Gravitational Quantities

Let us review the computation of gravitational quantities from geopotential spherical harmonic models. The disturbing potential $T$ can be expressed as (Rapp, 1982, p. 2)

$$T(r,\theta,\lambda) = \frac{GM}{r}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=2}^{n}(\overline{C}^*_{nm}\cos m\lambda + +\overline{S}_{nm}\sin m\lambda)\,\overline{P}_{nm}(\cos\theta), \tag{18}$$

where $GM$ is the geocentric gravitational constant, $r$ is the geocentric radius, $a$ stands for the equatorial radius of the mean earth's ellipsoid and $\overline{C}^*_{nm}$ is the difference between the actual coefficients $\overline{C}_{nm}$ and those implied by the reference equipotential ellipsoid $\overline{C}^u_{nm}$. Because of the rotational symmetry of the mean earth's ellipsoid, there will be only zonal terms. And because of the symmetry with respect to the equatorial plane, there will be only even zonal harmonics $\overline{C}^u_{2n,m}$ (Heiskanen and Moritz, 1967, p. 72). Then one may write the following relation $\overline{C}^*_{nm}$:

$$\begin{aligned} \overline{C}^*_n &= \overline{C}_n - \overline{C}^u_n \quad \text{if } m = 0, \\ \overline{C}^*_{nm} &= \overline{C}_{nm} \quad\quad\;\; \text{if } m \neq 0. \end{aligned} \tag{19}$$

The gravity anomaly $\Delta g$ can be expressed by (Torge, 1980, p. 155)

$$\Delta g\,(r,\theta,\lambda) = -\frac{\delta T}{\delta r} + \frac{1}{\gamma}\frac{\delta\gamma}{\delta r}T(r,\theta,\lambda), \tag{20}$$

where $\gamma$ is the normal gravity. Using the spherical approximation, we may write (Heiskanen and Moritz, 1967, p. 87)

$$\frac{1}{\gamma}\,\frac{\delta\gamma}{\delta r} = -\frac{2}{r}. \tag{21}$$

Then (20) becomes

$$\Delta g\,(r,\theta,\lambda) = -\frac{\delta T}{\delta r} - \frac{2}{r}T(r,\theta,\lambda). \tag{22}$$

Inserting (18) into (22]), one may write the following expression for computing the gravity anomalies from the geopotential coefficients

$$\Delta g\,(r,\theta,\lambda) = \frac{GM}{r^2}\sum_{n=2}^{\infty}(n-1)\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\overline{C}^*_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda)\,\overline{P}_{nm}(\cos\theta). \tag{23}$$

The height anomaly $\zeta$ can be given by the generalized Bruns formula as (Moritz, 1980, p. 353)

$$\zeta(r,\theta,\lambda) = \frac{T(r,\theta,\lambda)}{\gamma} \tag{24}$$

Inserting (18) into (24) gives

$$\zeta(r,\theta,\lambda) = \frac{GM}{\gamma r}\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n}(\overline{C}^*_{nm}\cos m\lambda + \overline{S}_{nm}\sin m\lambda)\,\overline{P}_{nm}(\cos\theta). \tag{25}$$

It should be noted that (24) can also be used for the calculation of the geoid undulation $N$ but with the evaluation of $T$ on the surface of the geoid by an appropriate choice of $r$.

## 4. The Proposed Technique: Field on Ellipsoid

Colombo's (1981) technique (described above) can be used for computing the harmonic coefficients of a non-scaled field defined on the sphere using HARMIN subroutine. In practical geodetic applications the situation is quite different. Let us consider a field of gravitational observables (e.g., gravity anomalies) defined on the mean earth's ellipsoid and it is required to compute the nondimensional potential coefficients. As mentioned before there is no direct mathematical relationship to transform the point or/and mean gravitational observables from the surface of the mean earth's ellipsoid to the surface of the sphere. Accordingly Colombo's technique cannot be used directly to compute the required nondimensional potential coefficients.

The developed HRCOFITR program can be used if the point/mean data field is defined either on sphere or on ellipsoid. The data field may be a non-scaled field, a gravity anomaly field or a geoid undulation field. In this section, we will consider data filed on the ellipsoid. In the next section, data field on the sphere will be considered. The case of non-scaled field will not be given here in detail. It is left for the reader as a small exercise.

Let us consider that we have a field defined on the ellipsoid. The main idea of the proposed technique is as follows. Assume, wrongly, that the field is defined on the sphere, then compute the harmonic coefficients. These harmonic coefficients are considered as an approximation to the correct ones. Hence, compute the field on the ellipsoid, and compute the residual field (which is also on the ellipsoid). Assume, again wrongly, that the residual field is defined on the sphere to compute the residual harmonic coefficients. Add the residual harmonic coefficients to the previously obtained values of the harmonic coefficients, and use these new coefficients to compute the field, and hence the residual field, on the ellipsoid. Repeat this process iteratively till two successive steps give practically the same harmonic coefficients.

Let us express the details of the proposed technique in the following steps:

1. Let us consider a field of gravitational observables (e.g., gravity anomalies) defined on the mean earth's ellipsoid. Assume, wrongly, that the data field is defined on the sphere. Using the modified HARMIN subroutine ((7) and (10)) compute the initial values of the non-scaled harmonic coefficients $\overline{A}^{\circ}_{nm}$, $\overline{B}^{\circ}_{nm}$. Note that within the modified HARMIN subroutine the computation of the Legendre functions (or their integrals in case of mean blocks data field) is done within the subroutine (instead of reading it from a file as the original version written by Colombo). This allows both very fast computation and saving of the disk space. The Legendre functions (or their integrals) are computed on the surface of the sphere, where the needed polar distance $\theta$ is computed by

$$\theta = 90^\circ - \phi. \tag{26}$$

2. Scale the computed non-scaled harmonic coefficients $\overline{A}^{\circ}_{nm}$, $\overline{B}^{\circ}_{nm}$ depending on the data field characteristic to compute the nondimensional potential harmonic coefficients $\overline{C}^{\circ}_{nm}$, $\overline{S}^{\circ}_{nm}$. If the data field $l$ were gravity anomalies, the scaling is done using the following expression (see (23))

$$\left\{ \begin{matrix} \overline{C}^{\circ}_{nm} \\ \overline{S}^{\circ}_{nm} \end{matrix} \right\} = \frac{R^2}{GM} \frac{1}{n-1} \left(\frac{R}{a}\right)^n \left\{ \begin{matrix} \overline{A}^{\circ}_{nm} \\ \overline{B}^{\circ}_{nm} \end{matrix} \right\}, \tag{27}$$

where $R$ is the radius of the mean earth's sphere. If the data field $l$ were geoid undulations, the scaling is done using the following expression (cf. (25))

$$\left\{ \begin{matrix} \overline{C}^{\circ}_{nm} \\ \overline{S}^{\circ}_{nm} \end{matrix} \right\} = \frac{\gamma R}{GM} \left(\frac{R}{a}\right)^n \left\{ \begin{matrix} \overline{A}^{\circ}_{nm} \\ \overline{B}^{\circ}_{nm} \end{matrix} \right\}, \tag{28}$$

where $\gamma$ is a mean value of normal gravity.

3. Use the nondimensional potential harmonic coefficients $\overline{C}^{\circ}_{nm}$, $\overline{S}^{\circ}_{nm}$ computed in the last step to create the computed field $\tilde{l}$ on the ellipsoid by the modified subroutine SSYNTH using (15) and (16). Note that within the modified SSYNTH subroutine the computation of the Legendre functions (or their integrals in case of mean blocks data field) is done within the subroutine (instead of reading it from a file as the original version written by Colombo). This again allows both very fast computation and saving of the disk space. The Legendre functions (or their integrals) are computed on the surface of the ellipsoid, where the needed polar distance $\theta$ is computed by

$$\theta = 90^\circ - \psi, \tag{29}$$

where $\psi$ is the geocentric latitude given by (Torge, 1980, p. 50)

$$\psi = \arctan\lfloor(1-e^2)\tan\phi\rfloor, \tag{30}$$

and $e$ is the first eccentricity of the mean earth's ellipsoid. The modified subroutine SSYNTH automatically scales the computed field according to its characteristic. For gravity anomalies data field, the scaled field $\hat{l}$ is computed from the non-scaled $\tilde{l}$, computed by the original expressions (15) and (16), using the following expression (cf. (23))

$$\hat{l} = \frac{GM}{r^2}(n-1)\left(\frac{a}{r}\right)^n \tilde{l}, \tag{31}$$

where $r$ is the geocentric radius, given by

$$r = a\sqrt{\frac{1-e^2(2-e^2)\sin^2\phi}{1-e^2\sin^2\phi}}. \tag{32}$$

For the geoid undulation data field the scaled field $\hat{l}$ is computed from the non-scaled field $\tilde{l}$ using the following expression (cf. (25))

$$\hat{l} = \frac{GM}{\gamma_\circ r}\left(\frac{a}{r}\right)^n \tilde{l}. \tag{33}$$

where $\gamma_\circ$ is the normal gravity on the surface of the ellipsoid given by (Heiskanen and Moritz, p. 76):

$$\gamma_\circ = \gamma_e \frac{1 + k\sin^2\phi}{\sqrt{1-e^2\sin^2\phi},} \tag{34}$$

where $k$ is given by

$$k = \frac{(1-f)\,\gamma_p - \gamma_e}{\gamma_e}. \tag{35}$$

Here $\gamma_e$ and $\gamma_p$ stand for the normal gravity at the equator and the pole, respectively, and $f$ stands for the flattening of the mean earth's ellipsoid.

4. Compute the residual field $\delta\, l$, which is already on the surface of the ellipsoid, by

$$\delta\, l = l - \hat{l}. \tag{36}$$

5. To assure good and quick convergence of the iterative solution, it is recommended to scale the residual field $\delta\, l$ obtaining a scaled residual field $\Delta\, l$ by the expression

$$\Delta\, l = \left(\frac{r}{a}\right)^p \delta\, l, \tag{37}$$

   where $p$ is a factor computed empirically in such a way that it is decreased if the solution converges slowly. The factor $p$ ranges between 2 $N_{max}$, with an initial value of $p = N_{max}$.

6. Assume that the scaled residual field is defined on the sphere. Compute the residual non-scaled harmonic coefficients $\delta\,\overline{A}_{nm}$, $\delta\,\overline{B}_{nm}$ using the modified HARMIN subroutine.

7. Scale the computed non-scaled residual harmonic coefficients $\delta\,\overline{A}_{nm}$, $\delta\,\overline{B}_{nm}$ depending on the data field characteristic to compute the nondimensional potential harmonic coefficients $\delta\,\overline{C}_{nm}$, $\delta\,\overline{S}_{nm}$. For gravity anomalies data field, the scaling is done using the following expression (see (23))

$$\left\{\begin{array}{c}\delta\,\overline{C}_{nm}\\ \delta\,\overline{S}_{nm}\end{array}\right\} = \frac{R^2}{GM}\frac{1}{n-1}\left(\frac{R}{a}\right)^n \left\{\begin{array}{c}\delta\,\overline{A}_{nm}\\ \delta\,\overline{B}_{nm}\end{array}\right\}. \tag{38}$$

   For geoid undulations data field, the scaling is done using the following expression (cf. (25))

$$\left\{\begin{array}{c}\delta\,\overline{C}_{nm}\\ \delta\,\overline{S}_{nm}\end{array}\right\} = \frac{\gamma R}{GM}\left(\frac{R}{a}\right)^n \left\{\begin{array}{c}\delta\,\overline{A}_{nm}\\ \delta\,\overline{B}_{nm}\end{array}\right\}. \tag{39}$$

8. Compute the nondimensional potential harmonic coefficients $\overline{C}_{nm}$, $\overline{S}_{nm}$ by

$$\left\{\begin{array}{c}\overline{C}_{nm}\\ \overline{S}_{nm}\end{array}\right\} = \left\{\begin{array}{c}\overline{C}^{\circ}_{nm}\\ \overline{S}^{\circ}_{nm}\end{array}\right\} + \left\{\begin{array}{c}\delta\,\overline{C}_{nm}\\ \delta\,\overline{S}_{nm}\end{array}\right\}. \tag{40}$$

9. Use the nondimensional potential harmonic coefficients $\overline{C}_{nm}$, $\overline{S}_{nm}$, computed in the last step, to create a computed scaled field $\hat{l}$ (same as step 3).

10. Repeat the steps 4, 5, 6 and 7.

11. Compute the nondimensional potential harmonic coefficients $\overline{C}_{nm}$, $\overline{S}_{nm}$ of the $i$-th iteration using their values in the preceding iteration and the residual nondimensional harmonic coefficients $\delta\,\overline{C}_{nm}$, $\delta\,\overline{S}_{nm}$ (computed from step 7) by

$$\left\{\begin{array}{c}\overline{C}_{nm}\\ \overline{S}_{nm}\end{array}\right\}_i = \left\{\begin{array}{c}\overline{C}_{nm}\\ \overline{S}_{nm}\end{array}\right\}_{i-1} + \left\{\begin{array}{c}\delta\,\overline{C}_{nm}\\ \delta\,\overline{S}_{nm}\end{array}\right\}, \tag{41}$$

   where the subscripts $i$ and $i-1$ refer to the iteration steps.

12. Repeat the steps 9, 10 and 11 until two successive iteration steps give practically the same harmonic coefficients, or alternatively, no practical change in the residual field between two successive iteration steps is happened.

## 5. The Proposed Technique: Field on Sphere

Let us consider that we have a field defined on the sphere. In the following, we list the steps of the developed technique in this case.

1. Consider a field of gravitational observables $l$ (e.g., gravity anomalies) defined on the sphere. Using the modified HARMIN subroutine ((7) and (10)) compute the initial values of the non-scaled harmonic coefficients $\overline{A}^{\circ}_{nm}$, $\overline{B}^{\circ}_{nm}$. The Legendre functions (or their integrals) are computed on the surface of the sphere, where the needed polar distance $\theta$ is computed by

$$\theta = 90^{\circ} - \phi. \tag{42}$$

   It should be noted that $\overline{A}^{\circ}_{nm}$, $\overline{B}^{\circ}_{nm}$ represent the output of Colombo's (1981) original HARMIN subroutine.

2. Scale the computed non-scaled harmonic coefficients $\overline{A}^{\circ}_{nm}$, $\overline{B}^{\circ}_{nm}$ depending on the data field characteristic to compute the nondimensional potential harmonic coefficients $\overline{C}^{\circ}_{nm}$, $\overline{S}^{\circ}_{nm}$. If the data field $l$ were gravity anomalies, the scaling is done using the following expression (see (23))

$$\left\{\begin{array}{c}\overline{C}^{\circ}_{nm}\\ \overline{S}^{\circ}_{nm}\end{array}\right\} = \frac{R^2}{GM}\frac{1}{n-1}\left\{\begin{array}{c}\overline{A}^{\circ}_{nm}\\ \overline{B}^{\circ}_{nm}\end{array}\right\}. \tag{43}$$

   If the data field $l$ were geoid undulations, the scaling is done using the following expression (cf. (25))

$$\left\{\begin{array}{c}\overline{C}^{\circ}_{nm}\\ \overline{S}^{\circ}_{nm}\end{array}\right\} = \frac{\gamma R}{GM}\left\{\begin{array}{c}\overline{A}^{\circ}_{nm}\\ \overline{B}^{\circ}_{nm}\end{array}\right\}. \tag{44}$$

   The coefficients $\overline{C}^{\circ}_{nm}$, $\overline{S}^{\circ}_{nm}$ represent the *scaled* coefficients computed by Colombo's technique.

3. Use the nondimensional potential harmonic coefficients $\overline{C}^{\circ}_{nm}$, $\overline{S}^{\circ}_{nm}$ computed in the last step to create the computed field $\tilde{l}$ on the sphere by the modified subroutine SSYNTH using (15) and (16). The Legendre functions (or their integrals) are computed on the surface of the

sphere, where the needed polar distance is computed again by

$$\theta = 90^\circ - \phi. \tag{45}$$

The modified subroutine SSYNTH automatically scales the computed field according to its characteristic. For gravity anomalies data field, the scaled field $\hat{l}$ is computed from the non-scaled field $\tilde{l}$, computed by the original expressions (15) and (16), using the following expression (cf. (23)).

$$\hat{l} = \frac{GM}{R^2}(n-1)\,\tilde{l}. \tag{46}$$

For the geoid undulation data field the scaled field $\hat{l}$ is computed from the non-scaled field $\tilde{l}$ using the following expression (cf. (25)).

$$\hat{l} = \frac{GM}{\gamma R}\,\tilde{l}. \tag{47}$$

4. Compute the residual field $\delta\, l$, which is already on the surface of the sphere, by

$$\delta\, l = l - \hat{l}. \tag{48}$$

Generally, according to the approximations involved in the FFT technique, the computed field $\hat{l}$ does not coincide with the data field $l$, and hence the residual field $\delta\, l$ is not zero. This is the reason of the iterative process in the case of data field on the sphere.

5. Compute the residual non-scaled harmonic coefficients $\delta\,\overline{A}_{nm}$, $\delta\,\overline{B}_{nm}$ corresponding to the residual field $\delta\, l$ using the modified HARMIN subroutine.

6. Scale the computed non-scaled residual harmonic coefficients $\delta\,\overline{A}_{nm}$, $\delta\,\overline{B}_{nm}$ depending on the data field characteristic to compute the nondimensional potential harmonic coefficients $\delta\,\overline{C}_{nm}$, $\delta\,\overline{S}_{nm}$. For gravity anomalies data field, the scaling is done using the following expression (see (23))

$$\begin{Bmatrix} \delta\,\overline{C}_{nm} \\ \delta\,\overline{S}_{nm} \end{Bmatrix} = \frac{R^2}{GM}\frac{1}{n-1}\begin{Bmatrix} \delta\,\overline{A}_{nm} \\ \delta\,\overline{B}_{nm} \end{Bmatrix}. \tag{49}$$

For geoid undulations data field, the scaling is done using the following expression (cf. (25))

$$\begin{Bmatrix} \delta\,\overline{C}_{nm} \\ \delta\,\overline{S}_{nm} \end{Bmatrix} = \frac{\gamma R}{GM}\begin{Bmatrix} \delta\,\overline{A}_{nm} \\ \delta\,\overline{B}_{nm} \end{Bmatrix}. \tag{50}$$

7. Compute the nondimensional potential harmonic coefficients $\overline{C}_{nm}$, $\overline{S}_{nm}$ by

$$\begin{Bmatrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{Bmatrix} = \begin{Bmatrix} \overline{C}^{\circ}_{nm} \\ \overline{S}^{\circ}_{nm} \end{Bmatrix} + \begin{Bmatrix} \delta\,\overline{C}_{nm} \\ \delta\,\overline{S}_{nm} \end{Bmatrix}. \tag{51}$$

8. Use the nondimensional potential harmonic coefficients $\overline{C}_{nm}$, $\overline{S}_{nm}$, computed in the last step, to create a computed scaled field $\hat{l}$ (same as step 3).

9. Repeat the steps 4, 5 and 6.

10. Compute the nondimensional potential harmonic coefficients $\overline{C}_{nm}$, $\overline{S}_{nm}$ of the $i$-th iteration using their values in the preceding iteration and the residual nondimensional harmonic coefficients $\delta\,\overline{C}_{nm}$, $\delta\,\overline{S}_{nm}$ (computed from step 6) by

$$\begin{Bmatrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{Bmatrix}_i = \begin{Bmatrix} \overline{C}_{nm} \\ \overline{S}_{nm} \end{Bmatrix}_{i-1} + \begin{Bmatrix} \delta\,\overline{C}_{nm} \\ \delta\,\overline{S}_{nm} \end{Bmatrix}, \tag{52}$$

where the subscripts $i$ and $i-1$ refer to the iteration steps.

11. Repeat the steps 8, 9 and 10 until two successive iteration steps give practically the same harmonic coefficients, or alternatively, no practical change in the residual field between two successive iteration steps has happened.

## 6. Numerical Tests

In order to examine the proposed technique, two computational tests were carried out. In the first test, the OSU91A geopotential model (Rapp et al., 1991) complete to degree and order 360 has been used to create two 30'×30' global data fields on the surface of the ellipsoid using the modified SSYNTH subroutine. The first field represents point gravity anomalies and the second field represents mean geoid undulations. The developed technique (HRCOFITR program) as well as Colombo's technique (modified HARMNIN subroutine) have been used to compute the nondimensional potential harmonic coefficients complete to degree and order 360 for both data fields.

Figure 1 shows the difference between the OSU91A coefficients and the coefficients computed by the developed technique for the point gravity anomaly data field defined on the ellipsoid. The solution needed 88 iterations performed in 37 minutes on a Pentium 300 MHz PC. It should be noteed that the estimated zonal harmonics are defined in the zero tide system. Except the even zonal harmonics (which have errors in the order of $10^{-12}$ – $10^{-11}$) and the higher degree harmonics ($n > 300$) till order 120 (which have errors ranging between $10^{-15}$ and $10^{-11}$), all other harmonics have practically zero error. This shows that the developed technique computed the harmonic coefficients with a relatively high accuracy. The residual field (data field created by OSU91A coefficients minus computed field created by estimated coefficients) ranges *only* between $-0.74$ mgal to 0.79 mgal with a zero mean and a standard deviation of 0.05 mgal (practically no residuals).

Figure 2 shows the difference between the OSU91A coefficients and the coefficients computed by Colombo's technique for the point gravity anomaly data field defined on the ellipsoid. Figure 2 shows that all harmonic coefficients have significantly great errors having its maximum value in the order of $10^{-8}$ for the lower degrees decreasing slowly with the degree to a value in the order of $10^{-13}$ for higher degrees. The residual field ranges between –29 mgal to 15 mgal with a zero mean and a standard deviation of 1.8 mgal. This comes directly from the fact that Colombo's technique assumes that the field is defined on the *sphere,* which signalizes that Colombo's technique is not adequate to analyze data fields defined on the ellipsoid.

Figure 3 shows the difference between the OSU91A coefficients and the coefficients computed by the developed technique for the mean geoid undulation data field defined on the ellipsoid. The solution needed 172 iterations performed in 2.5 hours on Pentium 300 MHz PC. Except the even zonal harmonics for $n > 30$ (which have errors in the order of $10^{-15}$ – $10^{-12}$) and the higher degree harmonics ($n > 260$) till order 60 (which have errors ranging between $10^{-15}$ and $10^{-12}$), all other harmonics have practically zero error. The residual field ranges *only* between $-0.71$ cm to 0.79 cm with a zero mean and a standard deviation of 0.04 cm (practically no residuals). This shows again the capability of the developed technique for estimating high accurate harmonic coefficients for fields defined on the surface of the ellipsoid.

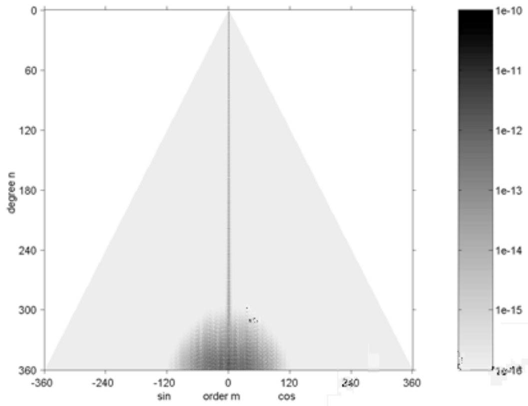


*Fig. 1. Difference between OSU91A coefficients and the coefficients computed by the developed technique for the point gravity anomaly data field defined on the ellipsoid.*

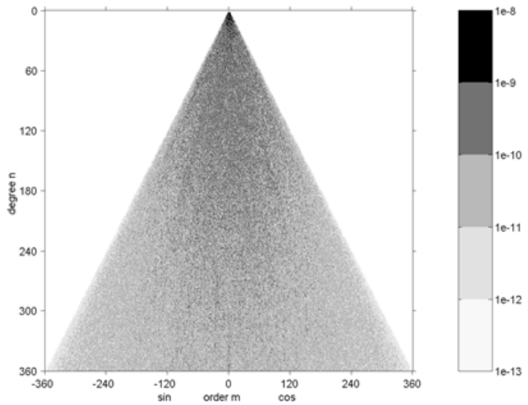


*Fig. 2. Difference between OSU91A coefficients and the coefficients computed by Colombo's technique for the point gravity anomaly data field defined on the ellipsoid*

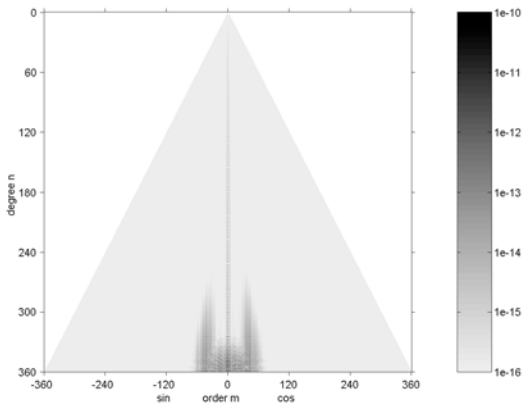


*Fig. 3. Difference between OSU91A coefficients and the coefficients computed by the developed technique for the mean geoid undulation data field defined on the ellipsoid.*

It should be noted that a similar conclusion for using Colombo's technique to estimate the harmonic coefficients for the mean geoid undulation data field has been made. The difference between the OSU91A coefficients and the coefficients computed by Colombo's technique for the mean geoid undulation data field shows a graph completely similar to Fig. 2.

In the second test, the OSU91A geopotential model complete to degree and order 360 has been used to create a 30'×30' global point gravity anomaly data field on the surface of the mean earth's sphere using the modified SSYNTH subroutine. The developed technique (HRCOFITR program) as well as Colombo's technique (modified HARMNIN subroutine) have then been used to compute the nondimensional potential harmonic coefficients complete to degree and order 360.

Figure 4 shows the difference between the OSU91A coefficients and the coefficients computed by Colombo's technique for the point gravity anomaly data field defined on the sphere. Figure 4 shows that only the zonal harmonics with a band of non-zonal harmonics, increases smoothly till $m = 50$ at $n = 360$, have significantly higher errors (ranging between $10^{-15}$ and $10^{-10}$). All other harmonics have practically zero error. The residual field ranges between $-6.78$ mgal to 4.55 mgal with a zero mean and a standard deviation of 0.43 mgal. The significantly high errors appearing here are due to the approximations involved in the FFT technique.

Figure 5 shows the difference between the OSU91A coefficients and the coefficients computed by the developed technique for the point gravity anomaly data field defined on the sphere. The solution needed 120 iterations performed in 40 minutes on a Pentium 300 MHz PC. Figure 5 shows that only the zonal harmonics with a very narrow band of non-zonal harmonics, increases linearly till $m = 12$ at $n = 360$, have slightly higher errors (ranging between $10^{-15}$ and $10^{-11}$). All other harmonic coefficients have practically zero error. The residual field ranges *only* between $-0.41$ mgal to 0.41 mgal with a zero mean and a standard deviation of 0.08 mgal (practically no residuals). This shows that using the developed technique improves the accuracy of the estimated harmonic coefficients even if the field is defined on the sphere.

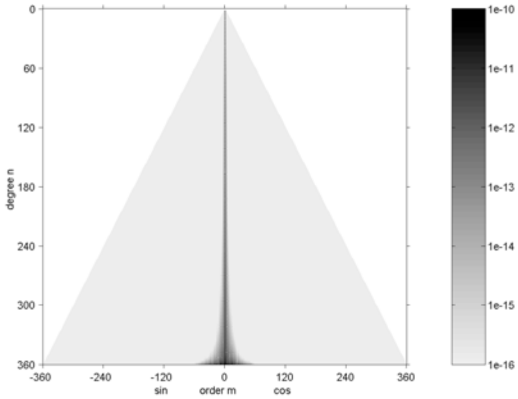


*Fig. 4. Difference between OSU91A coefficients and the coefficients computed by Colombo's technique for the point gravity anomaly data field defined on the sphere.*

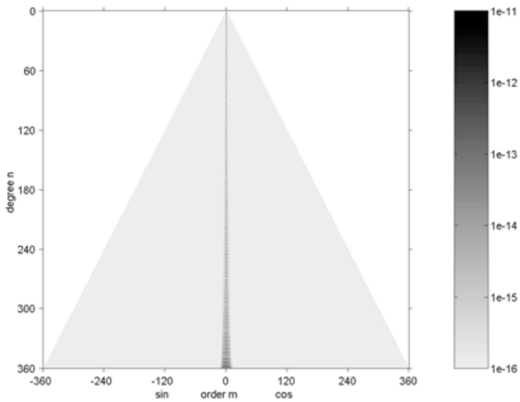


*Fig. 5. Difference between OSU91A coefficients and the coefficients computed by the developed technique for the point gravity anomaly data field defined on the sphere.*

## 7. Conclusion

The paper presents an efficient technique for harmonic analysis on a spheroid (both the sphere and the ellipsoid). The main idea of the proposed technique, implemented in the HRCOFITR program, is performed using Colombo's (1981) main subroutines HARMIN and SSYNTH (using FFT technique for harmonic analysis and synthesis on the sphere), after significant and critical modifications implemented by the author, in an iterative and scaling process for harmonic analysis on both the sphere and the ellipsoid. In order to examine the developed technique, two computational tests have been carried out. In the first test, two data fields on the ellipsoid have been created using

OSU91A geopotential model. In the second test a data field on the sphere has been created using OSU91A geopotential model. In all cases, the harmonic coefficients have been analyzed using Colombo's technique as well as using the developed technique. The results proved that the developed technique gives always better accuracy for the estimated harmonic coefficients as well as for the residual field. The results also show that using the developed technique improves the accuracy of the estimated harmonic coefficients even if the field is defined on the sphere. This is due to the approximations involved in the FFT technique used in Colombo's technique.

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